

HOMOGENEOUS SCHRÖDINGER OPERATORS ON HALFLINE

LAURENT BRUNEAU

JAN DEREZIŃSKI

VLADIMIR GEORGESCU

Let U_τ be the group of dilations on $L^2[0, \infty[$, that is $(U_\tau f)(x) = e^{\tau/2} f(e^\tau x)$. We say that A is **homogeneous of degree ν** if $U_\tau A U_\tau^{-1} = e^{-\nu\tau} A$.

Consider the differential expression

$$-\partial_x^2 + cx^{-2} \tag{1}$$

on $C_c^\infty(0, \infty)$. Clearly (1) is homogeneous of degree -2 .

(1) is essentially self-adjoint on $C_c^\infty]0, \infty[$ for $c \geq \frac{3}{4}$ and not essentially self-adjoint for $c < \frac{3}{4}$.

Theorem 1 *There is a unique **holomorphic family** $\{H_m\}_{\operatorname{Re} m > -1}$ such that H_m coincides with the closure of $-\partial_x^2 + (m^2 - \frac{1}{4})x^{-2}$ if $m \geq 1$.*

*For each m with $\operatorname{Re} m > -1$, the operators H_m are **homogeneous of degree -2** and satisfy $H_m^* = H_{\bar{m}}$, the **spectrum** of H_m is absolutely continuous and is equal to $[0, \infty[$.*

H_m is self-adjoint if m is real.

*If $0 \leq m < 1$, the operator H_m is the **Friedrichs extension** of $-\partial_x^2 + (m^2 - \frac{1}{4})x^{-2}$.*

*If $-1 < m \leq 0$, the operator H_m is the **Krein extension** of $-\partial_x^2 + (m^2 - \frac{1}{4})x^{-2}$.*

The operators H_m are interesting for many reasons.

- They have several interesting physical applications, eg. they appear in the decomposition of the **Aharonov-Bohm Hamiltonian**.
- They have rather subtle and rich properties, illustrating various concepts of the operator theory in Hilbert spaces (eg. the Friedrichs and Krein extensions, holomorphic families of closed operators).
- Surprisingly rich is also the theory of the first order homogeneous operators $A_\alpha = i\partial_x + i\frac{\alpha}{x}$, which is closely related to the theory of H_m .
- Essentially all basic objects related to H_m , such as their resolvents, spectral projections, wave and scattering operators, can be explicitly computed.
- A number of nontrivial identities involving special functions, especially the **Bessel functions**, find an appealing operator-theoretical interpretation in terms of the operators H_m . Eg. the **Barnes identity** leads to the formula for wave operators.

For each complex number α let \tilde{A}_α be the differential expression

$$\tilde{A}_\alpha := -i\partial_x + i\frac{\alpha}{x}$$

acting on distributions on \mathbb{R}_+ . Its restriction to $C_c^\infty]0, \infty[$ is a closable operator in $L^2]0, \infty[$ whose closure will be denoted A_α^{\min} . This is the **minimal operator** associated to \tilde{A}_α . The **maximal operator** A_α^{\max} associated to \tilde{A}_α is defined as the restriction of \tilde{A}_α to $\mathcal{D}(A_\alpha^{\max}) := \{f \in L^2]0, \infty[\mid \tilde{A}_\alpha f \in L^2]0, \infty[\}$.

The following properties of the operators A_α^{\min} and A_α^{\max} are easy to check:

- (i) $A_\alpha^{\min} \subset A_\alpha^{\max}$,
- (ii) $(A_\alpha^{\min})^* = A_{-\bar{\alpha}}^{\max}$ and $(A_\alpha^{\max})^* = A_{-\bar{\alpha}}^{\min}$,
- (iii) A_α^{\min} and A_α^{\max} are homogeneous of degree -1 .

Proposition 2 1. We have $A_\alpha^{\min} = A_\alpha^{\max}$ if and only if $|\operatorname{Re} \alpha| \geq 1/2$.

2. Let $\operatorname{Re} \alpha > -1/2$. Then

(i) $\operatorname{rs}(A_\alpha^{\max}) = \mathbb{C}_-$.

(ii) The map $\alpha \mapsto A_\alpha^{\max}$ is holomorphic in the region $\operatorname{Re} \alpha > -1/2$.

(iii) If $\operatorname{Re} \alpha \geq 0$ then iA_α^{\max} is the generator of a C^0 -semigroup of contractions

3. Let $\operatorname{Re} \alpha < 1/2$. Then

(i) $\operatorname{rs}(A_\alpha^{\min}) = \mathbb{C}_+$.

(ii) The map $\alpha \mapsto A_\alpha^{\min}$ is holomorphic in the region $\operatorname{Re} \alpha < 1/2$.

(iii) if $\operatorname{Re} \alpha \leq 0$ the operator $-iA_\alpha^{\min}$ is the generator of a C^0 -semigroup of contractions

For an arbitrary complex number m we introduce the differential expression

$$\tilde{L}_m := -\partial_x^2 + \frac{m^2 - 1/4}{x^2}$$

acting on distributions on \mathbb{R}_+ . Note that $\tilde{L}_m = \tilde{L}_{-m}$. We will however see that m , not m^2 , is the natural parameter.

Let L_m^{\min} and L_m^{\max} be the minimal and maximal operators associated to it in $L^2(0, \infty)$. It is clear that they are homogeneous operators of degree -2 and

$$(L_m^{\min})^* = L_{\bar{m}}^{\max}.$$

If m is a complex number we set

$$\zeta_m(x) = x^{1/2+m}.$$

Note that $\zeta_{\pm m}$ are both square integrable at the origin if and only if $|\operatorname{Re} m| < 1$.

We also choose $\xi \in C^\infty(\mathbb{R}_+)$ such that $\xi = 1$ on $[0, 1]$ and 0 on $[2, \infty[$.

For $\operatorname{Re}(m) > -1$, we define H_m to be the operator L_m^{\max} restricted to $\mathcal{D}(L_m^{\min}) + \mathbb{C}\xi\zeta_m$.

Theorem 3 For any $\operatorname{Re}(m) > -1$ we have $\operatorname{sp}(H_m) = [0, \infty[$. If $R_m(\lambda; x, y)$ is the integral kernel of the operator $(\lambda - H_m)$, then for $\operatorname{Re} k > 0$ we have

$$R_m(-k^2; x, y) = \begin{cases} \sqrt{xy} I_m(kx) K_m(ky) & \text{if } x < y, \\ \sqrt{xy} I_m(ky) K_m(kx) & \text{if } x > y. \end{cases},$$

where I_m is the *modified Bessel function* and K_m is the *MacDonald function*.

Theorem 4 Let $m > -1$. Then the operators H_m are positive, self-adjoint, homogeneous of degree 2 with $\text{sp } H_m = [0, \infty[$. Besides we have the following table:

$$m \geq 1: \quad H_m = A_{1/2+m}^* A_{1/2+m} = A_{1/2-m}^* A_{1/2-m}, \quad H_0^1 = \mathcal{Q}(H_m),$$

$$H_m = L_m^{\min} = L_m^{\max};$$

$$0 < m < 1: \quad H_m = A_{1/2+m}^* A_{1/2+m} = \left(A_{1/2-m}^{\min} \right)^* A_{1/2-m}^{\min} \quad H_0^1 = \mathcal{Q}(H_m),$$

H_m is the Friedrichs ext. of L_m^{\min} ;

$$m = 0: \quad H_0 = A_{1/2}^* A_{1/2}, \quad H_0^1 + c\xi\zeta_0 \text{ dense in } \mathcal{Q}(H_0),$$

H_0 is the Friedrichs and Krein ext. of L_0^{\min} ;

$$-1 < m < 0: \quad H_m = \left(A_{1/2+m}^{\max} \right)^* A_{1/2+m}^{\max}, \quad H_0^1 + c\xi\zeta_m = \mathcal{Q}(H_m),$$

H_m is the Krein ext. of L_m^{\min} .

In the region $-1 < m < 1$ (which is the most interesting one), it is quite remarkable that for strictly positive m one can factorize H_m in two different ways, whereas for $m \leq 0$ only one factorization appears.

As an example, let us consider the case of the Laplacian $-\partial_x^2$, i.e. $m^2 = 1/4$. The operators $H_{1/2}$ and $H_{-1/2}$ coincide with the **Dirichlet and Neumann Laplacian** respectively. One usually factorizes them as $H_{1/2} = P_{\min}^* P_{\min}$ and $H_{-1/2} = P_{\max}^* P_{\max}$, where P_{\min} and P_{\max} denote the usual momentum operator on the half-line with domain $\mathcal{H}_0^1[0, \infty[$ and $H^1[0, \infty[$ respectively. The above analysis says that, whereas for the Neumann Laplacian this is the only factorization of the form S^*S with S homogeneous, in the case of the Dirichlet Laplacian one can also factorize it in the rather unusual following way

$$H_{1/2} = (P_{\min} + ix^{-1})^* (P_{\min} + ix^{-1}).$$

Proposition 5 *The family H_m has the following property:*

$$0 \leq m \leq m' \Rightarrow H_m \leq H_{m'},$$

$$0 \leq m < 1 \Rightarrow H_{-m} \leq H_m.$$

Proposition 6 *Let $m \neq 0$.*

i) If $0 \leq \arg m \leq \pi/2$, then $\text{Num}(H_m) = \{z \mid 0 \leq \arg z \leq 2 \arg m\}$. Hence H_m is maximal sectorial and iH_m is dissipative.

ii) If $-\pi/2 \leq \arg m \leq 0$, then $\text{Num}(H_m) = \{z \mid 2 \arg m \leq \arg z \leq 0\}$. Hence H_m is maximal sectorial and $-iH_m$ is dissipative.

iii) If $|\arg m| \leq \pi/4$, then $-H_m$ is dissipative.

iv) If $\pi/2 < |\arg m| < \pi$, then $\text{Num}(H_m) = \mathbb{C}$.

Proposition 7 For $0 < a < b < \infty$, the integral kernel of $\mathbb{1}_{[a,b]}(H_m)$ is

$$\mathbb{1}_{[a,b]}(H_m)(x, y) = \int_{\sqrt{a}}^{\sqrt{b}} \sqrt{xy} J_m(kx) J_m(ky) k dk,$$

where J_m is the Bessel function.

Let \mathcal{F}_m be the operator on $L^2(0, \infty)$ given by

$$\mathcal{F}_m : f(x) \mapsto \int_0^\infty J_m(kx) \sqrt{kx} f(x) dx \quad (2)$$

Up to an inessential factor, \mathcal{F}_m is the so-called **Hankel transformation**.

Theorem 8 \mathcal{F}_m is a unitary involution on $L^2(0, \infty)$ diagonalizing H_m , more precisely

$$\mathcal{F}_m H_m \mathcal{F}_m^{-1} = x^2.$$

It satisfies $\mathcal{F}_m e^{itD} = e^{-itD} \mathcal{F}_m$ for all $t \in \mathbb{R}$.

Theorem 9 *If $m, k > -1$ are real then the **wave operators** associated to the pair H_m, H_k exist and*

$$\begin{aligned}\Omega_{m,k}^{\pm} &:= \lim_{t \rightarrow \pm\infty} e^{itH_m} e^{-itH_k} = e^{\pm i(m-k)\pi/2} \mathcal{F}_m \mathcal{F}_k \\ &= e^{\pm i(m-k)\pi/2} \frac{\Xi_k(D)}{\Xi_m(D)}.\end{aligned}\tag{3}$$

where

$$\Xi_m(t) = e^{i \ln(2)t} \frac{\Gamma\left(\frac{m+1+it}{2}\right)}{\Gamma\left(\frac{m+1-it}{2}\right)}.$$

The **scattering operator** $S_{m,k}$ for the pair (H_m, H_k) is a scalar operator $S_{m,k} = e^{i\pi(m-k)} \mathbb{1}$.

(3) has been obtained independently by Richard and Pankrashkin.

The definition (or actually a number of equivalent definitions) of a **holomorphic family of bounded operators** is quite obvious and does not need to be recalled. In the case of unbounded operators the situation is more subtle.

Suppose that Θ is an open subset of \mathbb{C} , \mathcal{H} is a Banach space, and $\Theta \ni z \mapsto H(z)$ is a function whose values are closed operators on \mathcal{H} . We say that this is a **holomorphic family of closed operators** if for each $z_0 \in \Theta$ there exists a neighborhood Θ_0 of z_0 , a Banach space \mathcal{K} and a holomorphic family of bounded operators $\Theta_0 \ni z \mapsto A(z) \in B(\mathcal{K}, \mathcal{H})$ such that $\text{Ran } A(z) = \mathcal{D}(H(z))$ and

$$\Theta_0 \ni z \mapsto H(z)A(z) \in B(\mathcal{K}, \mathcal{H})$$

is a holomorphic family of bounded operators.

We have the following practical criterion:

Theorem 10 *Suppose that $\{H(z)\}_{z \in \Theta}$ is a function whose values are closed operators on \mathcal{H} . Suppose in addition that for any $z \in \Theta$ the resolvent set of $H(z)$ is nonempty. Then $z \mapsto H(z)$ is a **holomorphic family of closed operators** if and only if for any $z_0 \in \Theta$ there exists $\lambda \in \mathbb{C}$ and a neighborhood Θ_0 of z_0 such that $\lambda \in \text{rs}(H(z))$ for $z \in \Theta_0$ and $z \mapsto (H(z) - \lambda)^{-1} \in B(\mathcal{H})$ is holomorphic on Θ_0 .*

The above theorem indicates that it is more difficult to study holomorphic families of closed operators that for some values of the complex parameter have an **empty resolvent set**.

It is interesting to note that $\Xi_m(D)$ is a unitary operator for all real values of m and

$$\Xi_m^{-1}(D)x^{-2}\Xi_m(D) \tag{4}$$

is a function with values in self-adjoint operators for all real m . $\Xi_m(D)$ is bounded and invertible also for all m such that $\operatorname{Re} m \neq -1, -2, \dots$. Therefore, the formula (4) defines an operator for all $\{m \mid \operatorname{Re} m \neq -1, -2, \dots\} \cup \mathbb{R}$. Clearly, for $\operatorname{Re} m > -1$, this operator function coincides with the operator H_m studied in this paper. Its spectrum is always equal to $[0, \infty[$ and it is analytic in the interior of its domain.

One can then pose the following question: does this operator function extend to a holomorphic function of closed operators **on the whole complex plane?**

A recent manuscript of M. Wrochna gives an affirmative answer to this question.