Unbounded linear operators

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Contents

These lecture notes are a continuation of the notes *"Bounded operators"*. We develop the theory of unbounded operators in Banach, and especially Hilbert spaces. We avoid using more advanced tools such as locally convex topologies and applications of the Baire category theorem.

1 Unbounded operators

1.1 Relations

Let X, Y be sets. R is called a relation iff $R \subset Y \times X$. We will also write $R : X \to Y$. (Note the inversion of the direction). An example of a relation is the identity $1_X := \{(x,x) \ : \ x \in X\} \subset X \times X$.

Introduce the "projections"

$$Y \times X \ni (y,x) \mapsto \pi_Y(y,x) := y \in Y,$$

$$Y \times X \ni (y,x) \mapsto \pi_X(y,x) := x \in X,$$

and the "flip"

$$Y \times X \ni (y,x) \mapsto \tau(y,x) := (x,y) \in X \times Y.$$

The domain of R is defined as $\mathrm{Dom} R := \pi_X R$, its range is $\mathrm{Ran}\, R = \pi_Y R$, the inverse of R is defined as $R^{-1} := \tau R \subset X \times Y$. If $S \subset Z \times Y$, then the superposition of S and R is defined as $S \circ R \subset Z \times X$, $S \circ R := \{(z,x) \in Z \times X \ : \ \underset{y \in Y}{\vee} \ (z,y) \in S, \ (y,x) \in R\}$.

If $X_0 \subset X$, then the restriction of R to X_0 is defined as

$$R\Big|_{X_0} := R \cap Y \times X_0.$$

If, moreover, $Y_0 \subset Y$, then

$$R\Big|_{X_0 \to Y_0} := R \cap Y_0 {\times} X_0.$$

We say that a relation R is right-unique, if for any $x \in X$ $\pi_Y(R \cap Y{\times}\{x\})$ is one-element. We say that R has a maximal domain if $\mathrm{Dom} R = X$.

Proposition 1.1 *a) If R, S are right-unique, then so is $S \circ R$.*
b) If R, S have a maximal domain, then so does $S \circ R$.

A right unique relation will be also called a pseudo-transformation (-operator, etc). Instead of writing $(y, x) \in R$, we will then write $y = R(x)$ or, in some contexts, $y = Rx$. We also introduce the graph of R:

$$\mathrm{Gr}\, R := \{(y, x) \in Y \times X \ : \ y = R(x), \ x \in \mathrm{Dom} R\}.$$

Note that strictly speaking $\mathrm{Gr}\, R = R$. The difference of $\mathrm{Gr}\, R$ and R lies only in their syntactic role.

Note that a superposition of pseudotransformations is a pseudotransformation.

We say that a pseudotransformation is injective if it is left-unique. The inverse of a pseudotransformation is a pseudotransformation iff it is injective.

A transformation is a pseudotransformation with a maximal domain. The composition of transformations is a transformation.

We say that a transformation R is bijective iff it is left-unique and $\mathrm{Ran}\, R = Y$. The inverse of a transformation is a transformation iff it is bijective.

Proposition 1.2 *Let $R \subset X{\times}Y$ and $S \subset Y{\times}X$ be transformations such that $R{\circ}S = 1_Y$ and $S{\circ}R = 1_X$. Then S and R are bijections and $S = R^{-1}$.*

1.2 Linear pseudooperators

Let $\mathcal{X}, \mathcal{Y}$ be vector spaces.

Proposition 1.3 *1) A linear subspace $\mathcal{V} \subset \mathcal{Y} \oplus \mathcal{X}$ is a graph of a certain pseudooperator iff $(y, 0) \in \mathcal{V}$ implies $y = 0$.*
2) A pseudooperator A is injective iff $(0, x) \in \mathrm{Gr}\, A$ implies $x = 0$.

From now on by an "operator" we will mean a "pseudooperator". To say that A is a true operator we will write $\mathrm{Dom} A = \mathcal{X}$.

1.3 Closed operators

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces.

Theorem 1.4 *Let $A : \mathcal{X} \to \mathcal{Y}$ be an operator. The following conditions are equivalent:*

(1) $\mathrm{Gr}\, A$ *is closed in* $\mathcal{Y} \times \mathcal{X}$.

(2) *If* $x_n \to x, \ x_n \in \mathrm{Dom} A$ *and* $Ax_n \to y$, *then* $x \in \mathrm{Dom} A$ *and* $y = Ax$.

(3) *For some* $p \in [1, \infty]$, $\mathrm{Dom} A$ *with the norm*

$$\|x\|_{A,p} := (\|Ax\|^p + \|x\|^p)^{\frac{1}{p}}.$$

is a Banach space.

Proof. The equivalence of (1), (2) and (3) is obvious, if we note that

$$\mathrm{Dom}A \ni x \mapsto (Ax, x) \in \mathrm{Gr}\, A$$

is a bijection. □

Definition 1.5 *An operator satisfying the above conditions is called closed.*

We will say that $A : \mathcal{X} \to \mathcal{Y}$ is bounded iff

$$\|Ax\| \leq c\|x\|, \quad x \in \mathrm{Dom}A. \tag{1.1}$$

$B(\mathcal{X}, \mathcal{Y})$ will denote all bounded operators from $\mathcal{X}$ to $\mathcal{Y}$ with the domain equal to $\mathcal{X}$.

Proposition 1.6 *A bounded operator A is closed iff $\mathrm{Dom}A$ is closed.*

Theorem 1.7 *Let $B \in B(\mathcal{X}, \mathcal{Y})$ be invertible and $A : \mathcal{X} \to \mathcal{Y}$ closed. Then BA is closed on $\mathrm{Dom}A$ and AB is closed on $B^{-1}\mathrm{Dom}A$.*

Proof. We check that

$$\begin{aligned}
&\|x\|_A \leq \max(1, \|B^{-1}\|)\|x\|_{BA}, \\
&\|x\|_{BA} \leq \max(1, \|B\|)\|x\|_A, \\
&\|Bx\|_A \leq \max(1, \|B\|)\|x\|_{AB}, \\
&\|x\|_{AB} \leq \max(1, \|B^{-1}\|)\|Bx\|_A.
\end{aligned}$$

□

Theorem 1.8 *If A is closed and injective, then so is A^{-1}.*

Proof. The flip $\tau : \mathcal{Y} \times \mathcal{X} \to \mathcal{X} \times \mathcal{Y}$ is continuous. □

1.4 Closable operators

Theorem 1.9 *Let $A : \mathcal{X} \to \mathcal{Y}$ be an operator. The following conditions are equivalent:*

(1) *There exists a closed operator B such that $B \supset A$.*

(2) $(\mathrm{Gr}\, A)^{\mathrm{cl}}$ *is the graph of an operator.*

(3) $(y, 0) \in (\mathrm{Gr}\, A)^{\mathrm{cl}} \Rightarrow y = 0$.

(4) $(x_n) \subset \mathrm{Dom}A$, $x_n \to 0$, $Ax_n \to y$ *implies* $y = 0$.

Definition 1.10 *An operator A satisfying the conditions of Theorem 1.9 is called closable. If the conditions of Theorem 1.9 hold, then the operator whose graph equals $(\mathrm{Gr}\, A)^{\mathrm{cl}}$ is denoted by A^{cl} and called the closure of A.*

Proof of Theorem 1.9 To show (2)⇒(1) it suffices to take as B the operator A^{cl}. Let us show (1)⇒(2). Let B be a closed operator such that $A \subset B$. Then $(\mathrm{Gr}\, A)^{\mathrm{cl}} \subset (\mathrm{Gr}\, B)^{\mathrm{cl}} = \mathrm{Gr}\, B$. But $(y, 0) \in \mathrm{Gr}\, B \Rightarrow y = 0$, hence $(y, 0) \in (\mathrm{Gr}\, A)^{\mathrm{cl}} \Rightarrow y = 0$. Thus $(\mathrm{Gr}\, A)^{\mathrm{cl}}$ is the graph of an operator. □

As a by-product of the above proof, we obtain

Proposition 1.11 *If A is closable, B closed and $A \subset B$, then $A^{\mathrm{cl}} \subset B$.*

Proposition 1.12 *Let A be bounded. Then A is closable,* $\mathrm{Dom}A^{\mathrm{cl}} = (\mathrm{Dom}A)^{\mathrm{cl}}$ *and* $A^{\mathrm{cl}}x = \lim_n Ax_n$ *for* $x_n \underset{n\to\infty}{\to} x$, $x_n \in \mathrm{Dom}A$. *Besides, A^{cl} satisfies (1.1).*

Let A be a closed operator. We say that a linear subspace $\mathcal{D}$ is an essential domain for A iff $\mathcal{D}$ is dense in $\mathrm{Dom}A$ in the graph topology. In other words, $\mathcal{D}$ is an essential domain for A, if

$$\left(A\Big|_{\mathcal{D}}\right)^{\mathrm{cl}} = A.$$

Theorem 1.13 (1) *If $A \in B(\mathcal{X}, \mathcal{Y})$, then a linear subspace $\mathcal{D} \subset \mathcal{X}$ is an essential domain for A iff it is dense in $\mathcal{X}$ (in the usual topology).*

(2) *If A is closed, has a dense domain and $\mathcal{D}$ is its essential domain, then $\mathcal{D}$ is dense in $\mathcal{X}$.*

1.5 Perturbations of closed operators

Definition 1.14 *Let B, $A : \mathcal{X} \to \mathcal{Y}$. We say that B is bounded relatively to A iff* $\mathrm{Dom}A \subset \mathrm{Dom}B$ *and there exist constants a, b such that*

$$\|Bx\| \leq a\|Ax\| + b\|x\|, \quad x \in \mathrm{Dom}A. \tag{1.2}$$

The infimum of a satisfying (1.2) is called the A-bound of B. In other words: the A-bound of B equals

$$\inf_{c>0} \sup_{x \in \mathrm{Dom}A\backslash\{0\}} \frac{\|Bx\|}{\|Ax\| + c\|x\|}.$$

In particular, if B is bounded, then its A-bound equals 0.

If A is unbounded, then its A-bound equals 1.

In the case of Hilbert spaces it is more convenient to use the following condition to define the relative boundedness:

Theorem 1.15 *B is bounded relatively to A with the A-bound a iff* $\mathrm{Dom}A \subset \mathrm{Dom}B$ *and*

$$\inf_{c>0} \sup_{x \in \mathrm{Dom}A\backslash\{0\}} \left(\frac{\|Bx\|^2}{\|Ax\|^2 + c\|x\|^2}\right)^{1/2} < \infty. \tag{1.3}$$

If this is the case, then (1.3) equals the A-bound of B

Proof. For any $\epsilon > 0$ we have

$$\begin{aligned} &\left(\|Ax\|^2 + c^2\|x\|^2\right)^{\frac{1}{2}} \\ &\leq \|Ax\| + c\|x\| \\ &\leq \left((1+\epsilon^2)\|Ax\|^2 + c^2(1+\epsilon^{-2})\|x\|^2\right)^{\frac{1}{2}}. \end{aligned}$$

□

Theorem 1.16 *Let A be closed and let B be bounded relatively to A with the A-bound less than* 1. *Then $A + B$ with the domain* $\mathrm{Dom}A$ *is closed. All essential domains of A are essential domains of $A + B$*

Proof. We know that

$$\|Bx\| \leq a\|Ax\| + b\|x\|$$

for some $a < 1$ and b. Hence

$$\|(A+B)x\| + \|x\| \leq (1+a)\|Ax\| + (1+b)\|x\|$$

and

$$(1-a)\|Ax\| + \|x\| \leq \|Ax\| - \|Bx\| + (1+b)\|x\| \leq \|(A+B)x\| + (1+b)\|x\|.$$

Hence the norms $\|Ax\| + \|x\|$ and $\|(A+B)x\| + \|x\|$ are equivalent on $\mathrm{Dom}A$. □

Theorem 1.17 *Suppose that A, C are two operators with the same domain $\mathrm{Dom}A = \mathrm{Dom}C = \mathcal{D}$ satisfying*

$$\|(A-C)x\| \leq a(\|Ax\| + \|Cx\|) + b\|x\|$$

for some $a < 1$. Then

(1) *A is closed on $\mathcal{D}$ iff C is closed on $\mathcal{D}$.*

(2) *A is closable on $\mathcal{D}$ iff C is closable on $\mathcal{D}$ and then the domains of A^{cl} and C^{cl} coincide.*

Proof. Define $B := C - A$ and $F(t) := A + tB$ with the domain $\mathcal{D}$. For $0 \leq t \leq 1$, we have

$$\begin{aligned} \|Bx\| &\leq a(\|Ax\| + \|Cx\|) + b\|x\| \\ &= a\left(\|(F(t) - tB)x\| + \|(F(t) + (1-t)B)x\|\right) + b\|x\| \\ &\leq 2a\|F(t)x\| + a\|Bx\| + b\|x\| \end{aligned}$$

Hence

$$\|Bx\| \leq \frac{2a}{1-a}\|F(t)x\| + \frac{b}{1-a}\|x\|.$$

Therefore, if $|s| < \frac{1-a}{2a}$ and $t, t+s \in [0,1]$, then $F(t+s)$ is closed iff $F(t)$ is closed. □

1.6 Invertible unbounded operators

Definition 1.18 *We say that an operator A is invertible iff $A^{-1} \in B(\mathcal{Y}, \mathcal{X})$.*

Theorem 1.19 *Let A be closed. Suppose that for some $c > 0$*

$$\|Ax\| \geq c\|x\|, \quad x \in \mathrm{Dom}A. \tag{1.4}$$

Then $\mathrm{Ran}\,A$ is closed. If $\mathrm{Ran}\,A = Y$, then A is invertible and

$$\|A^{-1}\| \leq c^{-1}$$

Proof. Let $y_n \in \mathrm{Ran}\,A$ and $y_n \to y$. Let $Ax_n = y_n$. Then x_n is a Cauchy sequence. Hence there exists $\lim_{n\to\infty} x_n := x$. But A is closed, hence $Ax = y$. Therefore, $\mathrm{Ran}\,A$ is closed. □

Corollary 1.20 *Let A be closed. Suppose that for some $c > 0$*

$$\|Ax\| \geq c\|x\|,$$

and $\mathrm{Ran}\,A$ is dense in $\mathcal{Y}$. Then A is invertible

Theorem 1.21 *Let A be closable and for some c (1.4) holds. Then (1.4) holds for $A^{\rm cl}$ as well.*

Theorem 1.22 (1) *Let A be injective and* $\mathrm{Dom}B \supset \mathrm{Dom}A$. *Then B has the A-bound less than $\|BA^{-1}\|$.*

(2) *If, moreover, $\|BA^{-1}\| < 1$, then $A + B$ with the domain* $\mathrm{Dom}A$ *is closed, invertible and*

$$(A+B)^{-1} = \sum_{j=0}^{\infty}(-1)^j A^{-1}(BA^{-1})^j.$$

Proof. Let $a := \|BA^{-1}\|$. By the estimate

$$\|Bx\| \leq a\|Ax\|, \quad x \in \mathrm{Dom}A,$$

we see that B has the A-bound less than or equal to a. This proves (1).

Assume now that $a < 1$. Let

$$C_n := \sum_{j=0}^{n}(-1)^j A^{-1}(BA^{-1})^j.$$

Then $\lim_{n\to\infty} C_n =: C$ exists.

Let $y \in \mathcal{Y}$. Clearly, $\lim_{n\to\infty} C_n y = Cy$.

$$(A+B)C_n y = y + (-1)^n (BA^{-1})^{n+1} y \to y.$$

But $A + B$ is closed, hence $Cy \in \mathrm{Dom}(A+B)$ and $(A+B)Cy = y$.

If $x \in \mathrm{Dom}(A+B)$, then

$$C_n(A+B)x = x + (-1)^n A^{-1}(BA^{-1})^n Ax \to x.$$

Hence $C(A+B)y = y$. □

Proposition 1.23 *Let A and B be invertible and* $\mathrm{Dom}B \supset \mathrm{Dom}A$. *Then*

$$B^{-1} - A^{-1} = B^{-1}(A-B)A^{-1}.$$

1.7 Spectrum of unbounded operators

Let A be an operator on $\mathcal{X}$. We define the resolvent set of A as

$$\mathrm{rs}A := \{z \in \mathbb{C} \ : \ z - A \ \text{ is invertible } \}.$$

We define the spectrum of A as $\mathrm{sp}A := \mathbb{C}\backslash\mathrm{rs}A$.

We say that $x \in \mathcal{X}$ is an eigenvector of A with the eigenvalue $\lambda \in \mathbb{C}$ iff $x \in \mathrm{Dom}A$, $x \neq 0$ and $Ax = \lambda x$. The set of eigenvalues is called the point spectrum of A and denoted $\mathrm{sp_p}A$. Clearly, $\mathrm{sp_p}A \subset \mathrm{sp}A$.

Let $\mathbb{C}^{\rm comp}$ denote the Riemann sphere (the one-point compactification of $\mathbb{C}$). In the case of unbounded operators it is sometimes convenient to use the "extended spectrum", which is a subset of $\mathbb{C}^{\rm comp}$, instead of the usual spectrum—a subset of $\mathbb{C}$.

The extended resolvent set is defined as $\mathrm{rs}^{\rm ext}A := \mathrm{rs}A \cup \{\infty\}$ if $A \in B(\mathcal{X})$ and $\mathrm{rs}^{\rm ext}A := \mathrm{rs}A$, if A is unbounded. The extended spectrum is defined as

$$\mathrm{sp}^{\rm ext}A = \mathbb{C}^{\rm comp}\backslash\mathrm{rs}^{\rm ext}A.$$

If $A \in B(\mathcal{X})$, we set $(\infty - A)^{-1} = 0$.

Theorem 1.24 (1) *If* $\lambda, \mu \in \mathrm{rs}A$*, then*

$$(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1}.$$

(2) *If* $\lambda \in \mathrm{rs}A$ *and* $\|(\lambda - A)^{-1}\| = c$*, then* $\{z :\ |z - \lambda| < c^{-1}\} \subset \mathrm{rs}A$.

(3) $\|(z - A)^{-1}\| \geq (\mathrm{dist}(z, \mathrm{sp}A))^{-1}$.

(4) *If* $\mathrm{rs}A$ *is nonempty, then* A *is closed.*

(5) $\mathrm{sp}^{\mathrm{ext}}A$ *is a compact subset of* $\mathbb{C}^{\mathrm{comp}}$.

(6) $(z - A)^{-1}$ *is analytic on* $\mathrm{rs}^{\mathrm{ext}}A$.

(7) $(z - A)^{-1}$ *cannot be analytically extended to a larger subset of* $\mathbb{C}^{\mathrm{comp}}$ *than* $\mathrm{rs}^{\mathrm{ext}}(A)$.

(8) $\mathrm{sp}^{\mathrm{ext}}(A) \neq \emptyset$

(9) $\mathrm{Ran}\,(z - A)^{-1}$ *does not depend on* $z \in \mathrm{rs}A$ *and equals* $\mathrm{Dom}A$.

(10) $\mathrm{Ker}(z - A)^{-1} = \{0\}$.

Proof. Let us show (4). If $\lambda \in \mathrm{rs}(A)$, then $\lambda - A$ is invertible, hence closed. □

It is easy to define the functional calculus for unbounded operators with a non-empty resolvent set. The definition usual definition can be repeated verbatim, replacing $\mathrm{sp}A$ with $\mathrm{sp}^{\mathrm{ext}}(A)$. Theorem **??** remains valid except that the point about convergent power series should be dropped

Proposition 1.25 *Suppose that* $\mathrm{rs}A$ *is non-empty and* $\mathrm{Dom}A$ *is dense. Then* $\mathrm{Dom}A^2$ *is dense.*

Proof. Let $z \in \mathrm{rs}A$. $(z - A)^{-1}$ is a bounded operator with a dense range and $\mathrm{Dom}A$ is dense. Hence $(z - A)^{-1}\mathrm{Dom}A$ is dense. If $x \in \mathrm{Dom}A$, then $A(z - A)^{-1}x = (z - A)^{-1}Ax \in \mathrm{Dom}A$ Hence $(z - A)^{-1}\mathrm{Dom}A \subset \mathrm{Dom}A^2$. □

Theorem 1.26 *Let* A *and* B *be operators on* $\mathcal{X}$ *with* $A \subset B$, $A \neq B$*. Then* $\mathrm{rs}A \subset \mathrm{sp}B$*, and hence* $\mathrm{rs}B \subset \mathrm{sp}A$.

Proof. Let $\lambda \in \mathrm{rs}A$. Let $x \in \mathrm{Dom}B\backslash\mathrm{Dom}A$. We have $\mathrm{Ran}\,(\lambda - A) = \mathcal{X}$, hence there exists $y \in \mathrm{Dom}A$ such that $(\lambda - A)y = (\lambda - B)x$. Hence $(\lambda - B)y = (\lambda - B)x$. Hence $\lambda \notin \mathrm{rs}B$. □

1.8 Examples of unbounded operators

Example 1.27 *Let* I *be an infinite set and* $(a_i)_{i\in I}$ *be an unbounded complex sequence. Let* $C_0(I)$ *be the space of sequences with a finite number of non-zero elements. For* $1 \leq p < \infty$ *we define the operator*

$$L^p(I) \supset C_0(I) \ni x \mapsto Ax \in L^p(I)$$

by the formula

$$(Ax)_i = a_i x_i.$$

(We can use $C_\infty(I)$ *instead of* $L^p(I)$*, then* $p = \infty$ *in the formulas below). Then the operator* A *is unbounded and non-closed. Besides,*

$$\mathrm{sp}_{\mathrm{p}}(A) = \{a_i\ :\ i \in I\},$$
$$\mathrm{sp}A = \mathbb{C}.$$

The closure of A has the domain

$$\mathrm{Dom}A^{\mathrm{cl}} := \{(x_i)_{i\in I} \in L^p(I) \ : \ \sum_{i\in I} |a_i x_i|^p < \infty\} \tag{1.5}$$

We then have

$$\mathrm{sp}_{\mathrm{p}}(A^{\mathrm{cl}}) = \{a_i \ : \ i \in I\},$$
$$\mathrm{sp}A^{\mathrm{cl}} = \{a_i \ : \ i \in I\}^{\mathrm{cl}}.$$

To prove this let $\mathcal{D}$ be the rhs of (1.5) and $x \in \mathcal{D}$. Then there exists a countable set I_1 such that $i \notin I_1$ implies $x_i = 0$. We enumerate the elements of I_1: $i_1, i_2, \dots$. Define $x^n \in C_0(I)$ setting $x^n_{i_j} = x_{i_j}$ for $j \leq n$ and $x^n_i = 0$ for the remaining indices. Then $\lim_{n\to\infty} x^n = x$ and $Ax^n \to Ax$. Hence, $\{(x, Ax) \ : \ x \in \mathcal{D}\} \subset (\mathrm{Gr}\, A)^{\mathrm{cl}}$.

If x^n belongs to (1.5) and $(x^n, Ax^n) \to (x, y)$, then $x^n_i \to x_i$ and $a_i x^n_i = (Ax^n)_i \to y_i$. Hence $y_i = a_i x_i$. Using that $y \in L^p(I)$ we see that x belongs to (1.5).

Example 1.28 *Let $p^{-1} + q^{-1} = 1$, $1 < p \leq \infty$ and let $(w_i)_{i\in I}$ be a sequence that does not belong to $L^q(I)$. Let $C_0(I)$ be as above. Define*

$$L^p(I) \supset C_0(I) \ni x \mapsto \langle w|x\rangle := \sum_{i\in I} x_i w_i \in \mathbb{C}.$$

Then $\langle w|$ is non-closable.

It is sufficient to assume that $I = \mathbb{N}$ and define $v^n_i := \frac{|w_i|^q}{w_i(\sum_{i=1}^n |w_i|^q)}$, $i \leq n$, $v^n_i = 0$, $i > n$. Then $\langle w|v^n\rangle = 1$ and $\|v^n\|_p = (\sum_{i=1}^n |w_i|^q)^{-\frac{1}{q}} \to 0$. Hence $(0, 1)$ belongs to the closure of the graph of the operator.

1.9 Pseudoresolvents

Definition 1.29 *Let $\Omega \subset \mathbb{C}$ be open. Then the continuous function*

$$\Omega \ni z \mapsto R(z) \in B(\mathcal{X})$$

is called a pseudoresolvent if

$$R(z_1) - R(z_2) = (z_2 - z_1) R(z_1) R(z_2). \tag{1.6}$$

Evidently, every resolvent of a closed operator is a pseudoresolvent.

Proposition 1.30 *Let $\Omega \ni z \mapsto R_n(z) \in B(\mathcal{X})$ be a sequence of pseudoresolvents and $R(z) := \mathrm{s-}\lim_{n\to\infty} R_n(z)$. Then $R(z)$ is a pseudoresolvent.*

Theorem 1.31 *Let $\Omega \ni z \mapsto R(z) \in B(\mathcal{X})$ be a pseudoresolvent. Then*

(1) *$\mathcal{R} := \mathrm{Ran}\, R(z)$ does not depend on $z \in \Omega$.*

(2) *$\mathcal{N} := \mathrm{Ker} R(z)$ does not depend on $z \in \Omega$.*

(3) *$R(z)$ is an analytic function and*

$$\frac{\mathrm{d}}{\mathrm{d}z} R(z) = -R(z)^2.$$

(4) *$R(z)$ is a resolvent of a certain operator iff $\mathcal{N} = \{0\}$. The domain of this operator equals $\mathcal{R}$.*

Proof. Let us prove (4)$\Leftarrow$. Fix $z_1 \in \Omega$. If $\mathcal{N} = \{0\}$, then every element of $\mathcal{R}$ can be uniquely represented as $R(z_1)x$, $x \in \mathcal{X}$. Define $HR(z_1)x := -x + z_1 R(z_1)x$. By formula (1.6) we check that the definition does not depend on z_1. □

2 One-parameter semigroups in Banach spaces

2.1 (M, β)-type semigroups

Let $\mathcal{X}$ be a Banach space.

Definition 2.1 $[0, \infty[\ni t \mapsto W(t) \in B(\mathcal{X})$ *is called a strongly continuous one-parameter semigroup iff*

(1) $W(0) = 1$;

(2) $W(t_1)W(t_2) = W(t_1 + t_2)$, $t_1, t_2 \in [0, \infty[$;

(3) $\lim_{t\downarrow 0} W(t)x = x$, $x \in \mathcal{X}$;

(4) *for some* $t_0 > 0$, $\|W(t)\| < M$, $0 \leq t \leq t_0$.

Remark 2.2 *Using the Banach-Steinhaus Theorem one can show that (4) follows from the remaining points.*

Theorem 2.3 *If* $W(t)$ *is a strongly continuous semigroup, then*

$$[0, \infty[\times X \ni (t, x) \mapsto W(t)x \in \mathcal{X}$$

is a continuous function. Besides, there exist constants M, β *such that*

$$\|W(t)\| \leq M\mathrm{e}^{\beta t}. \tag{2.7}$$

Proof. By (4), for $t \leq nt_0$ we have $\|W(t)\| \leq M^n$. Hence, $\|W(t)\| \leq M \exp(\frac{t}{t_0} \log M)$. Therefore, (2.7) is satisfied.

Let $t_n \to t$ and $x_n \to x$. Then

$$\begin{aligned}
\|W(t_n)x_n - W(t)x\| &\leq \|W(t_n)x_n - W(t_n)x\| + \|W(t_n)x - W(t)x\| \\
&\leq M\mathrm{e}^{\beta t_n}\|x_n - x\| + M\mathrm{e}^{\beta \min(t_n,t)}\|W(|t - t_n|)x - x\|.
\end{aligned}$$

□

We say that the semigroup $W(t)$ is (M, β)-type, if the condition (2.7) is satisfied.

Clearly, if $W(t)$ is (M, β)-type, then $W(t)\mathrm{e}^{-\beta t}$ is $(M, 0)$-type. Since $W(0) = 1$, no semigroups (M, β) exist for $M < 1$.

2.2 Generator of a semigroup

If $W(t)$ is a strongly continuous one-parameter semigroup, we define

$$\mathrm{Dom}A := \{x \in \mathcal{X} \ : \ \text{there exists } \lim_{t\to 0} t^{-1}(W(t)x - x)\}.$$

The operator A with the domain $\mathrm{Dom}A$ is defined by the formula

$$Ax := \lim_{t\to 0} t^{-1}(W(t)x - x).$$

A will be called the generator of $W(t)$.

Theorem 2.4 (1) *A is a closed densely defined operator;*

(2) $W(t)\mathrm{Dom}A \subset \mathrm{Dom}A$ *and* $W(t)A = AW(t)$;

(3) *If* $W_1(t)$ *and* $W_2(t)$ *are two different semigroups, then their generators are different.*

By (3), if A is an operator, which is a generator of a semigroup $W(t)$, then such $W(t)$ is unique. We will write $W(t) =: \mathrm{e}^{tA}$.

Proof of Theorem 2.4 (2). Let $x \in \mathrm{Dom}A$. Then

$$\lim_{s\downarrow 0} s^{-1}(W(s)-1)W(t)x = W(t)\lim_{s\downarrow 0} s^{-1}(W(s)-1)x = W(t)Ax. \tag{2.8}$$

Hence the limit of the left hand side of (2.8) exists. Hence $W(t)x \in \mathrm{Dom}A$ and $AW(t)x = W(t)Ax$. □

Lemma 2.5 *For* $x \in \mathcal{X}$ *put*

$$B_t x := t^{-1}\int_0^t W(s)x\mathrm{d}s.$$

Then

(1) $\mathrm{s}-\lim_{t\to 0} B_t = 1$.

(2) $B_t W(s) = W(s)B_t$.

(3) *If* $x \in \mathcal{X}$*, then* $B_t x \in \mathrm{Dom}A$,

$$AB_t x = t^{-1}(W(t)x - x), \tag{2.9}$$

(4) *For* $x \in \mathrm{Dom}A$*,* $AB_t x = B_t Ax$*.*

Proof. (1) follows by

$$B_t x - x = t^{-1}\int_0^t (W(s)x - x)\mathrm{d}s \underset{t\downarrow 0}{\to} 0.$$

(2) is obvious. (4) is proven as Theorem 2.4 (2). To prove (3) we note that

$$u^{-1}(W(u)-1)B_t x = t^{-1}(W(t)-1)B_u x \underset{u\downarrow 0}{\to} t^{-1}(W(t)x - x).$$

□

Proof of Theorem 2.4 (1), (3) The density of $\mathrm{Dom}A$ follows by Lemma 2.5 (1) and (3).

Let us show that A is closed. Let $x_n \underset{n\to\infty}{\to} x$ and $Ax_n \underset{n\to\infty}{\to} y$. Using the boundedness of $B_t A = AB_t$ we get

$$B_t y = \lim_{n\to\infty} B_t Ax_n = B_t Ax.$$

Hence

$$y = \lim_{t\downarrow 0} B_t y = \lim_{t\downarrow 0} B_t Ax = Ax.$$

□

Proposition 2.6 *Let* $W(t)$ *be a semigroup and* A *its generator. Then, for any* $x \in \mathrm{Dom}A$ *there exists a unique solution of*

$$[0,t_0] \ni t \mapsto x(t) \in \mathrm{Dom}A, \quad \frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t), \tag{2.10}$$

(for $t = 0$ *the derivative is right-sided). The solution is given by* $x(t) = W(t)x$*.*

Proof. Let us show that $x(t) := W(t)x$ solves (2.10). We already know that the right-sided derivative equals $Ax(t)$. It suffices to prove the same about the left-sided derivative. For $0 \le u \le t$ we have

$$(-u)^{-1}(W(t-u)x - W(t)x) = W(t-u)u^{-1}(W(u)-1)x \to_{u\to 0} W(t)Ax = AW(t)x.$$

Let us show now the uniqueness. Let $x(t)$ solve (2.10) . Let $y(s) := W(t-s)x(s)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}s}y(s) = W(t-s)Ax(s) - AW(t-s)x(s) = 0$$

Hence $y(s)$ does not depend on s. At $s = t$ it equals $x(t)$, and at $s = 0$ it equals $W(t)x$. □

Proof of Theorem 2.4 (3) By Theorem 2.6 (2), $W(t)$ is uniquely determined by A on Dom$A$. But $W(t)$ is bounded and Dom$A$ is dense, hence $W(t)$ is uniquely determined. □

2.3 Norm continuous semigroups

Theorem 2.7 (1) *If $A \in B(\mathcal{X})$, then $\mathbb{C} \ni z \mapsto \mathrm{e}^{zA}$ is a norm continuous group and A is its generator.*

(2) *If a one-parameter semigroup $W(t)$ is norm continuous, then its generator is bounded.*

Proof. (1) follows by the functional calculus.

Let us show (2). $W(t)$ is norm continuous, hence $\lim_{t\to 0} B_t = 1$. Therefore, for $0 < t < t_0$

$$\|B_t - 1\| < 1.$$

Hence B_t is then invertible.

We know that for $x \in \mathrm{Dom}A$

$$t^{-1}(W(t) - 1)x = B_t Ax.$$

For $0 \le t < t_0$ we can write this as

$$Ax = t^{-1}B_t^{-1}(W(t) - 1)x.$$

Hence $\|Ax\| \le c\|x\|$. □

2.4 Essential domains of generators

Theorem 2.8 *Let $W(t)$ be a strongly continuous one-parameter semigroup and let A be its generator. Let $\mathcal{D} \subset$ Dom$A$ be dense in $\mathcal{X}$ and $W(t)\mathcal{D} \subset \mathcal{D}$, $t > 0$. Then $\mathcal{D}$ is dense in* Dom$A$ *in the graph topology—in other words, $\mathcal{D}$ is an essential domain of A.*

Lemma 2.9 (1) *For $x \in \mathcal{X}$, $\|B_t x\|_{\mathrm{Dom}A} \le (Ct^{-1} + 1)\|x\|$;*

(2) *For $x \in \mathrm{Dom}A$, $\lim_{t\downarrow 0} \|B_t x - x\|_{\mathrm{Dom}A} = 0$;*

(3) *$W(t)$ is a strongly continuous semi-group on* Dom$A$ *equipped with the graph norm.*

(4) *If $\tilde{\mathcal{D}}$ is a closed subspace in* Dom$A$ *invariant wrt $W(t)$, then it is invariant also wrt B_t.*

Proof. (1) follows by Lemma 2.5 (3).

(2) follows by Lemma 2.5 (1) and because $B(t)$ commutes with A.

(3) follows from the fact that $W(t)$ is a strongly continuous semigroup on $\mathcal{X}$, preserves Dom$A$ and commutes with A.

To show (4), note that $B_t x$ is defined using an integral involving $W(s)x$. $W(s)x$ depends continuously on s in the topology of Dom$A$, as follows by (3). Hence this integral (as Riemann's integral) is well defined. Besides, $B_t x$ belongs to the closure of the space spanned by $W(s)x$, $0 \le s \le t$. □

Proof of Theorem 2.8. Let $x \in \mathrm{Dom}A$, $x_n \in \mathcal{D}$ and $x_n \underset{n\to\infty}{\to} x$ in $\mathcal{X}$. Let $\tilde{\mathcal{D}}$ be he closure of $\mathcal{D}$ in Dom$A$. Then $B_t x_n \in \tilde{\mathcal{D}}$, by Lemma 2.9 (4). By Lemma 2.9 (1) we have

$$\|B_t x_n - B_t x\|_{\mathrm{Dom}A} \le C_t\|x_n - x\|.$$

Hence $B_t x \in \tilde{\mathcal{D}}$. By Lemma 2.9 (2)

$$\|B_t x - x\|_{\mathrm{Dom}A} \underset{t\downarrow 0}{\to} 0.$$

Hence, $x \in \tilde{\mathcal{D}}$. □

2.5 Operators of (M, β)-type

Theorem 2.10 *Let A be a densely defned operator. Then the following conditions are equivalent:*

(1) $[\beta, \infty[\subset \mathrm{rs}(A)$ *and*

$$\|(x - A)^{-m}\| \leq M|x - \beta|^{-m}, \quad m = 1, 2, \dots, \quad x \in [\beta, \infty[$$

(2) $\{z \in \mathbb{C} \ : \ \mathrm{Re}z > \beta\} \subset \mathrm{rs}(A)$ *and*

$$\|(z - A)^{-m}\| \leq M|\mathrm{Re}z - \beta|^{-m}, \quad m = 1, 2, \dots, \quad z \in \{z \in \mathbb{C} \ : \ \mathrm{Re}z > \beta\}.$$

Proof. It suffices to prove (1)⇒(2). Let (1) be satisfied. It suffices to assume that $\beta = 0$. Let $z = x + \mathrm{i}y$. Then for $t > 0$

$$\begin{aligned} (z - A)^{-m} &= (x + t - A)^{m}(1 + (\mathrm{i}y - t)(x + t - A)^{-1})^{-m} \\ &= \sum_{j=0}^{\infty} (x + t - A)^{-m-j}(iy - t)^j \begin{pmatrix} -m \\ j \end{pmatrix}. \end{aligned}$$

Using the fact that $\begin{pmatrix} -m \\ j \end{pmatrix}$ has an alternating sign we get

$$\begin{aligned} \|(z - A)^{-m}\| &\leq \textstyle\sum_{j=0}^{\infty} |x + t|^{-m-j}(-1)^j |iy - t|^j \begin{pmatrix} -m \\ j \end{pmatrix} \\ &= M|x + t|^m \left(1 - \tfrac{|iy-t|}{x+t}\right)^{-m} \\ &= M(x + t - |iy - t|)^{-m} \underset{t\to\infty}{\to} Mx^{-m}. \end{aligned}$$

□

Definition 2.11 *We say that an operator A is (M, β)-type, iff the conditions of Theorem 2.10 are satisfied.*

Obviously, if A is of (M, β)-type, then $A - \beta$ is of $(M, 0)$-type.

2.6 The Hille-Philips-Yosida theorem

Theorem 2.12 *If $W(t)$ is a semigroup of (M, β)-type, then its generator A is also of (M, β)-type. Besides,*

$$(z - A)^{-1} = \int_0^{\infty} \mathrm{e}^{-tz} W(t) \mathrm{d}t, \quad \mathrm{Re}z > \beta.$$

Proof. Set

$$R(z)x := \int_0^\infty \mathrm{e}^{-zt} W(t)x\mathrm{d}t.$$

Let $y = R(z)x$. Then

$$\begin{aligned} & u^{-1}(W(u) - 1)y \\ & = -u^{-1}\mathrm{e}^{zu} \textstyle\int_0^u \mathrm{e}^{-zt} W(t)x\mathrm{d}t + u^{-1}(\mathrm{e}^{zu} - 1) \textstyle\int_0^\infty \mathrm{e}^{-zt} W(t)x\mathrm{d}t \underset{u\downarrow 0}{\rightarrow} -x + zy. \end{aligned}$$

Hence $y \in \mathrm{Dom}A$ and $(z - A)R(z)x = x$.

Suppose now that $x \in \mathrm{Ker}(z - A)$. Then $x_t := \mathrm{e}^{zt}x \in \mathrm{Dom}A$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t}x_t = Ax_t$. Hence $x_t = W(t)x$. But $\|x_t\| = \mathrm{e}^{\mathrm{Re}zt}\|x\|$, which is impossible.

By the formula

$$(z - A)^{-m} = \int_0^\infty \cdots \int_0^\infty \mathrm{e}^{-z(t_1+\cdots+t_m)} W(t_1 + \cdots + t_m)\mathrm{d}t_1 \cdots \mathrm{d}t_m$$

we get the estimate

$$\|(z - A)^{-m}\| \leq \int_0^\infty \cdots \int_0^\infty M\mathrm{e}^{-(z-\beta)(t_1+\cdots+t_m)}\mathrm{d}t_1 \cdots \mathrm{d}t_m = M|z - \beta|^{-m}.$$

□

Theorem 2.13 *If A is an operator of (M, β)-type, then it is the generator of a semigroup $W(t)$. This semigroup is of (M, β)-type.*

To simplify, let us assume that $\beta = 0$ (which does not restrict the generality). Then we have the formula

$$W(t) = \mathrm{s-}\lim_{n\to\infty} \left(1 - \frac{t}{n}A\right)^{-n},$$

$$\left\| W(t)x - \left(1 - \frac{t}{n}A\right)^{-n} x \right\| \leq M\frac{t^2}{2}\|A^2x\|, \quad x \in \mathrm{Dom}A^2.$$

Proof. Set

$$V_n(t) := \left(1 - \frac{t}{n}A\right)^{-n}.$$

Let us first show that

$$\mathrm{s-}\lim_{t\downarrow 0} V_n(t) = 1. \tag{2.11}$$

To prove (2.11) it suffices to prove that

$$\mathrm{s-}\lim_{s\downarrow 0}(1 - sA)^{-1} = 1. \tag{2.12}$$

We have $(1 - sA)^{-1} - 1 = (s^{-1} - A)^{-1}A$. Hence for $x \in \mathrm{Dom}A$

$$\|(1 - sA)^{-1}x - x\| \leq Ms^{-1}\|Ax\|,$$

which proves (2.12).

Let us list some other properties of $V_n(t)$: for $\mathrm{Re}t > 0$, $V_n(t)$ is holomorphic, $\|V_n(t)\| \leq M$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} V_n(t) = A\left(1 - \frac{t}{n}A\right)^{-n-1}.$$

To show that $V_n(t)x$ is a Cauchy sequence for $x \in \mathrm{Dom}(A^2)$, we compute

$$\begin{aligned}
V_n(t)x - V_m(t)x \quad &= \lim_{s\downarrow 0} V_n(t-s)V_m(s)x - \lim_{s\uparrow t} V_n(t-s)V_m(s)x \\
&= \lim_{\epsilon\downarrow 0} \int_\epsilon^{t-\epsilon} \tfrac{\mathrm{d}}{\mathrm{d}s} V_n(t-s)V_m(s)x \\
&= \lim_{\epsilon\downarrow 0} \int_\epsilon^{t-\epsilon} \Big(- V_n'(t-s)V_m(s) + V_n(t-s)V_m'(s)\Big)x \\
&= \lim_{\epsilon\downarrow 0} \int_\epsilon^{t-\epsilon} \left(\tfrac{s}{n} - \tfrac{t-s}{m}\right) \left(1 - \tfrac{t-s}{n}A\right)^{-n-1} \left(1 - \tfrac{s}{n}A\right)^{-m-1} A^2x.
\end{aligned}$$

Hence for $x \in \mathrm{Dom}(A^2)$

$$\begin{aligned}
\|V_n(t)x - V_m(t)x\| \quad &\leq \|A^2x\| \int_0^t |\tfrac{s}{m} - \tfrac{t-s}{n}| M^2 \mathrm{d}s \\
&= M^2(\tfrac{1}{n} + \tfrac{1}{m})\tfrac{t^2}{2}.
\end{aligned}$$

By the Proposition 1.25, $\mathrm{Dom}(A^2)$ is dense in $\mathcal{X}$. Therefore, there exists a limit uniform on $[0, t_0]$

$$\mathrm{s-}\lim_{n\to\infty} V_n(t) =: W(t),$$

which depends strongly continuously on t.

Finally, let us show that $W(t)$ is a semigroup with the generator A. To this end it suffices to show that for $x \in \mathrm{Dom}A$

$$\frac{\mathrm{d}}{\mathrm{d}t} W(t)x = AW(t)x. \tag{2.13}$$

But $x \in \mathrm{Dom}A$

$$V_n(t+u)x = V_n(t)x + \int_t^{t+u} A\Big(1 - \frac{s}{n}A\Big)^{-1} V_n(s)x\mathrm{d}s$$

Hence passing to the limit we get

$$W(t+u)x = W(t)x + \int_t^{t+u} AW(s)x\mathrm{d}s.$$

This implies (2.13). □

2.7 Semigroups of contractions and dissipative operators

Theorem 2.14 *Let A be a closed operator on $\mathcal{X}$. Then the following conditions are eqivalent:*

(1) *A is a generator of a semigroup of contractions, eg.* $\|\mathrm{e}^{tA}\| \leq 1$

(2) *The operator A is of $(1,0)$-type.*

(3) $]0, \infty[\subset \mathrm{rs}(A)$ *and*

$$\|(\mu - A)^{-1}\| \leq \mu^{-1}, \quad \mu \in \mathbb{R},\ \mu > 0,$$

(4) $\{z \in \mathbb{C} \ : \ \mathrm{Re}z > 0\} \subset \mathrm{rs}(A)$ *and*

$$\|(z - A)^{-1}\| \leq |\mathrm{Re}z|^{-1}, \quad z \in \mathbb{C},\ \mathrm{Re}z > 0.$$

(5) $\{z\in\mathbb{C}\ :\ \mathrm{Re}z>0\}\subset \mathrm{rs}A$*; besides, if* $x\in\mathrm{Dom}A$, $v\in\mathcal{X}^\#$, $\|v\|=1$ *and* $\langle v|x\rangle=\|x\|$, *then*

$$\mathrm{Re}\langle v|Ax\rangle\leq 0.$$

(6) *There exists* $z\in\mathbb{C}$ *with* $\mathrm{Re}z>0$ *such that* $\mathrm{Ran}\,(z-A)=X$*; besides if* $x\in\mathrm{Dom}A$, *then there* $\vee\, v\in X^\#$ *such that* $\|v\|=1$, $\langle v|x\rangle=\|x\|$, *and*

$$\mathrm{Re}\langle v|Ax\rangle\leq 0.$$

Proof. The equivalence of (1) and (2) is a special case of Theorems 2.12 and 2.13. The implications (2)⇒(3) and (2)⇒(4) are obvious, the converse implications are easy.

Let us show (1),(3)⇒(5). We have

$$\begin{aligned}&\mathrm{Re}\langle v|x\rangle=\langle v|x\rangle=\|x\|,\\ &\mathrm{Re}\,\langle v|\mathrm{e}^{tA}x\rangle\leq|\langle v|\mathrm{e}^{tA}x\rangle|\leq\|x\|.\end{aligned}$$

Hence

$$\mathrm{Re}\langle v|Ax\rangle=\lim_{t\downarrow 0}\mathrm{Re}t^{-1}(\langle v|\mathrm{e}^{tA}x\rangle-\langle v|x\rangle)\leq 0.$$

We know that if $\mathrm{Re}z>0$, then $z\in\mathrm{rs}(A)$. Hence $\mathrm{Ran}\,(z-A)=\mathcal{X}$.

The implication (5)⇒(6) is obvious.

Let us prove (6)⇒(3).

$$\begin{aligned}\|(z-A)x\|\quad&\geq|\langle v|(z-A)x\rangle|\\ &\geq\mathrm{Re}\langle v|(z-A)x\rangle\quad\geq\mathrm{Re}z\langle v|x\rangle=\|x\|\mathrm{Re}z.\end{aligned}$$

Using $\mathrm{Ran}\,(z-A)^{-1}=\mathcal{X}$, we conclude that $(z-A)^{-1}$ exists and $\|(z-A)^{-1}\|\leq|\mathrm{Re}z|^{-1}$. □

3 Unbounded operators in Hilbert spaces

3.1 Graph scalar product

Let $\mathcal{V}$, $\mathcal{W}$ be Hilbert spaces. Let $A:\mathcal{V}\to\mathcal{W}$ be an operator with domain $\mathrm{Dom}A$. It is natural to treat $\mathrm{Dom}A$ as a space with a scalar product

$$(v_1|v_2)_A:=(v_1|v_2)+(Av_1|Av_2).$$

Clearly, $\mathrm{Dom}A$ is a Hilbert space with this product iff A is closed.

3.2 The adjoint of an operator

Definition 3.1 *Let* $A:\mathcal{V}\to\mathcal{W}$ *have a dense domain. Then* $w\in\mathrm{Dom}A^*$, *iff the functional*

$$\mathrm{Dom}A\ni v\mapsto(w|Av)$$

is bounded (in the topology of $\mathcal{V}$*). Hence there exists a unique* $y\in\mathcal{V}$ *such that*

$$(w|Av)=(y|v),\quad v\in\mathcal{V}.$$

We set then

$$A^*w=y.$$

If $C_{\mathcal{V}}$, $C_{\mathcal{W}}$ are the Riesz antiisomorphisms, then we have the relationship between the (Banach space)-conjugate $A^{\#}$ and the (Hilbert space)-adjoint A^*:

$$A^* = C_{\mathcal{V}}^{-1} A^{\#} C_{\mathcal{W}}.$$

Theorem 3.2 *Let $A : \mathcal{V} \to \mathcal{W}$ have a dense domain. Then*

(1) *A^* is closed.*

(2) *$\mathrm{Dom} A^*$ is dense in $\mathcal{W}$ iff A is closable.*

(3) *$(\mathrm{Ran}\, A)^\perp = \mathrm{Ker} A^*$.*

(4) *$\mathrm{Dom} A \cap (\mathrm{Ran}\, A^*)^\perp \supset \mathrm{Ker} A$.*

Proof. Let $j : \mathcal{V} \oplus \mathcal{W} \to \mathcal{W} \oplus \mathcal{V}$, $j(v, w) := (-w, v)$. Note that j is unitary. We have

$$\mathrm{Gr}\, A^* = j(\mathrm{Gr}\, A)^\perp.$$

Hence $\mathrm{Gr}\, A^*$ is closed. This proves (1).

Let us prove (2).

$$\begin{aligned} w \in (\mathrm{Dom} A^*)^\perp &\Leftrightarrow (0, w) \in (\mathrm{Gr}\, A^*)^\perp \\ &\Leftrightarrow (w,) \in (\mathrm{Gr}\, A)^{\perp\perp} = (\mathrm{Gr}\, A)^{\mathrm{cl}}. \end{aligned}$$

Proof of (3):

$$\begin{aligned} w \in \mathrm{Ker} A^* &\Leftrightarrow (A^* w | v) = 0, \quad v \in \mathcal{V} \\ &\Leftrightarrow (A^* w | v) = 0, \quad v \in \mathrm{Dom} A \\ &\Leftrightarrow (w | A v) = 0, \quad v \in \mathrm{Dom} A \\ &\Leftrightarrow w \in (\mathrm{Ran}\, A)^\perp. \end{aligned}$$

Proof of (4)

$$\begin{aligned} v \in \mathrm{Ker} A &\Leftrightarrow (w | A v) = 0, \quad w \in \mathcal{W} \\ &\Rightarrow (w | A v) = 0, \quad w \in \mathrm{Dom} A^* \\ &\Leftrightarrow (A^* w | v) = 0, \quad w \in \mathrm{Dom} A^* \\ &\Leftrightarrow v \in (\mathrm{Ran}\, A^*)^\perp. \end{aligned}$$

Theorem 3.3 *Let $A : \mathcal{V} \to \mathcal{W}$ be closable with a dense domain. Then*

(1) *A^* is closed with a dense domain.*

(2) *$A^* = (A^{\mathrm{cl}})^*$.*

(3) *$(A^*)^* = A^{\mathrm{cl}}$*

(4) *$(\mathrm{Ran}\, A)^\perp = \mathrm{Ker} A^*$. Hence A^* is injective iff $\mathrm{Ran}\, A$ is dense.*

(5) *$(\mathrm{Ran}\, A^*)^\perp = \mathrm{Ker} A$. Hence A is injective iff $\mathrm{Ran}\, A^*$ is dense.*

Proof. (1) was proven in Theorem 3.2.

To see (2) note that

$$\mathrm{Gr}\, A^* = j(\mathrm{Gr}\, A)^\perp = j((\mathrm{Gr}\, A)^{\mathrm{cl}})^\perp = \mathrm{Gr}\, A^{\mathrm{cl}*}.$$

To see (3) we use

$$\mathrm{Gr}\ (A^{**}) = j^{-1} \left(j(\mathrm{Gr}\, A)^\perp \right)^\perp = (\mathrm{Gr}\, A)^{\perp\perp} = (\mathrm{Gr}\, A)^{\mathrm{cl}}.$$

(4) is proven in Theorem 3.2.

To prove (5) note that in the second line of the proof of Theorem 3.2 (4) we can use the fact that $\mathrm{Dom}A^*$ is dense in $\mathcal{W}$ to replace $\Rightarrow$ with $\Leftrightarrow$. □

3.3 Inverse of the adjoint operator

Theorem 3.4 *Let A be densely defined, closed, injective and with a dense range. Then*

(1) A^{-1} *is densely defined, closed, injective and with a dense range.*

(2) A^* *is densely defined, closed, injective and with a dense range.*

(3) $(A^*)^{-1} = (A^{-1})^*$.

Proof. (1) and (2) sum up previously proven facts.

To prove (3), recall the maps $\tau, j : \mathcal{V} \oplus \mathcal{W} \to \mathcal{W} \oplus \mathcal{V}$. We have

$$\mathrm{Gr}\, A^* = j(\mathrm{Gr}\, A)^{\perp}, \quad \mathrm{Gr}\, A^{-1} = \tau(\mathrm{Gr}\, A).$$

Hence

$$\mathrm{Gr}\, A^{-1*} = j(\tau(\mathrm{Gr}\, A))^{\perp} = \tau^{-1}(j(\mathrm{Gr}\, A)^{\perp}) = \mathrm{Gr}\, A^{*-1}.$$

□

Theorem 3.5 *Let $A : \mathcal{V} \to \mathcal{W}$ be densely defined and closed. Then the following conditions are equivalent:*

(1) *A is invertible.*

(2) *A^* is invertible.*

(3) *For some $c > 0$, $\|Av\| \geq c\|v\|$, $v \in \mathcal{V}$ and $\|A^*w\| \geq c\|v\|$, $w \in \mathcal{W}$.*

Moreover, $\mathrm{sp}^{\mathrm{ext}}(A) = \overline{\mathrm{sp}^{\mathrm{ext}}(A^*)}$.

Proof. (1)$\Rightarrow$(2). Let A be invertible. Then $A^{-1} \in B(\mathcal{W}, \mathcal{V})$. Hence, $A^{-1*} \in B(\mathcal{V}, \mathcal{W})$.

Clearly, the assumptions of Theorem 3.4 are satisfied, and hence $A^{*-1} = A^{-1*}$. Therefore, $A^{*-1} \in B(\mathcal{V}, \mathcal{W})$.

(1)$\Leftarrow$(2). A^* is also densely defined and closed. Hence the same arguments as above apply.

It is obvious that (1) and (2) imply (3). Let us prove that (3)$\Rightarrow$(1). $\|A^*v\| \geq c\|v\|$ implies that $\mathrm{Ker}A^* = \{0\}$. Hence $(\mathrm{Ran}\, A)^{\perp}$ is dense. This together with $\|Av\| \geq c\|v\|$ implies that $\mathrm{Ran}\, A = \mathcal{W}$, and consequently, A is invertible. □

3.4 Maximal operators

The numerical range of the operator T is defined as

$$\mathrm{Num}T = \{(v|Tv) \ : \ v \in \mathrm{Dom}T, \quad \|v\| = 1\}.$$

Theorem 3.6 (1) $\|(z - T)v\| \geq \mathrm{dist}(z, \mathrm{Num}T)\|v\|, \quad v \in \mathrm{Dom}T.$

(2) *If T is a closed operator and $z \in \mathbb{C}\backslash(\mathrm{Num}T)^{\mathrm{cl}}$, then $z - T$ has a closed range.*

(3) *If $z \in \mathrm{rs}T\backslash\mathrm{Num}T$, then $\|(z - T)^{-1}\| \leq |\mathrm{dist}(z, \mathrm{Num}T)|^{-1}$.*

(4) *Let Δ be a connected component of $\mathbb{C}\backslash(\mathrm{Num}T)^{\mathrm{cl}}$. Then either $\Delta \subset \mathrm{sp}T$ or $\Delta \subset \mathrm{rs}T$.*

Proof. To prove (1), take $(z_0 \not\in \mathrm{Num}T)^{\mathrm{cl}}$. Recall that $\mathrm{Num}T$ is convex. Hence, replacing T wih $\alpha T+\beta$ we can assume that $z_0=\mathrm{i}\nu$ and $0\in \mathrm{Num}T\subset\{\mathrm{Im}z\leq 0\}$. Thus $\nu=\mathrm{dist}(\mathrm{i}\nu,\mathrm{Num}T)$ and

$$\begin{aligned}\|(\mathrm{i}\nu-T)v\|^2 &= (Tv|Tv)-i\nu(v|Tv)+\mathrm{i}\nu(Tv|v)+|\nu|^2\|v\|^2\\ &= (Tv|Tv)-2\nu\mathrm{Im}(v|Tv)+|\nu|^2\|v\|^2\\ &\geq |\nu|^2\|v\|^2.\end{aligned}$$

(1) implies (2) and (3).

Let $z_0\in\mathrm{rs}T\backslash\mathrm{Num}T$. By (3), if $r=\mathrm{dist}(z_0,\mathrm{Num}T)$, then $\{|z-z_0|<r\}\subset\mathrm{rs}T$. This proves (4). □

Definition 3.7 *An operator T is called maximal, if* $\mathrm{sp}T\subset(\mathrm{Num}T)^{\mathrm{cl}}$.

Clearly, if T is a maximal operator, and $z\not\in(\mathrm{Num}T)^{\mathrm{cl}}$, then

$$\|(z-T)^{-1}\|\leq(\mathrm{dist}(z,\mathrm{Num}T)))^{-1}.$$

If T is bounded, then T is maximal.

Theorem 3.8 *Suppose that T is an operator and for any connected component Δ_i of $\mathbb{C}\backslash(\mathrm{Num}T)^{\mathrm{cl}}$ we choose $\lambda_i\in\Delta_i$. Then the following conditions are necessary and sufficient for T to be maximal*

1. *For all i, $\lambda_i\not\in\mathrm{sp}T$;*
2. *T is closable and for all i,* $\mathrm{Ran}\,(\lambda_i-T)=\mathcal{V}$.
3. *T is closed and for all i,* $\mathrm{Ran}\,(\lambda_i-T)$ *is dense in $\mathcal{V}$.*
4. *T is closed and for all i,* $\mathrm{Ker}(\overline{\lambda}_i-T^*)=\{0\}$.

3.5 Dissipative operators

We say that an operator A is dissipative iff

$$\mathrm{Im}(v|Av)\leq 0,\quad v\in\mathrm{Dom}A.$$

Equivalently, A is dissipative iff $\mathrm{Num}A\subset\{\mathrm{Im}z\leq 0\}$.

A is maximally dissipative iff A is dissipative and $\mathrm{sp}A\subset\{\mathrm{Im}z\leq 0\}$.

Theorem 3.9 *Let A be a densely defined operator. Then the following conditions are equivalent:*

1. *$-\mathrm{i}A$ is the generator of a strongly continuous semigroup of contractions.*
2. *A is maximally dissipative.*

Proof. (1) ⇒(2) We have

$$\mathrm{Re}(v|\mathrm{e}^{-\mathrm{i}tA}v)\leq|(v|\mathrm{e}^{-\mathrm{i}tA}v)|\leq\|v\|^2.$$

Hence

$$\begin{aligned}\mathrm{Im}(v|Av) &= -\mathrm{Re}\mathrm{i}(v|Av)\\ &= \mathrm{Re}\lim_{t\nearrow 0}t^{-1}\left((v|\mathrm{e}^{-\mathrm{i}tA}v)-\|v\|^2\right)\leq 0.\end{aligned}$$

Hence A is dissipative.

We know that the generators of contractions satisfy $\{\mathrm{Re}z>0\}\subset\mathrm{rs}(-\mathrm{i}A)$.

(2)⇒(1) Let $\mathrm{Re} z > 0$. We have

$$\begin{aligned} \|v\|\|(z+\mathrm{i}A)v\| &\geq |(v|(z+\mathrm{i}A)v)| \\ &\geq \mathrm{Re}(v|(z+\mathrm{i}A)v) \geq \mathrm{Re} z\|v\|^2. \end{aligned}$$

Hence, noting that $z \in \mathrm{rs}A$, we obtain $\|(z+\mathrm{i}A)^{-1}\| \leq \mathrm{Re} z^{-1}$. Therefore, A is an operator of the type $(1,0)$. □

Theorem 3.10 *Let A be dissipative. Then the following conditions are equivalent:*

(1) *A is maximally dissipative.*

(2) *A is closable and there exists z_0 with $\mathrm{Im} z_0 > 0$ and $\mathrm{Ran}\,(z_0 - A) = \mathcal{V}$.*

(3) *A is closed and there exists z_0 with $\mathrm{Im} z_0 > 0$ and $\mathrm{Ran}\,(z_0 - A)$ dense in $\mathcal{V}$.*

(4) *A is closed and there exists z_0 with $\mathrm{Im} z_0 > 0$ and $\mathrm{Ker}(\overline{z}_0 - A^*) = \{0\}$.*

3.6 Hermitian operators I

An operator $A : \mathcal{V} \to \mathcal{V}$ is hermitian iff

$$(Aw|v) = (w|Av), \ w, v \in \mathrm{Dom}A.$$

Clearly, A is hermitian iff $\mathrm{Num}A \subset \mathbb{R}$.

Remark 3.11 *In a part of literature the term "symmetric" is used instead of "hermitian".*

Theorem 3.12 *Let A be densely defined and hermitian. Then A is closable. Besides, one of the following possibilities is true:*

(1) $\mathrm{sp}A \subset \mathbb{R}$.

(2) $\mathrm{sp}A = \{\mathrm{Im} z \geq 0\}$.

(3) $\mathrm{sp}A = \{\mathrm{Im} z \leq 0\}$.

(4) $\mathrm{sp}A = \mathbb{C}$.

Proof. $A \subset A^*$ and A^* is closed. Hence A is closable. □

Theorem 3.13 *Let A be a densely defined operator. Then the following conditions are equivalent:*

(1) *$-\mathrm{i}A$ is the generator of a strongly continuous semigroup of isometries.*

(2) *A is hermitian and $\mathrm{sp}A \subset \{\mathrm{Im} z \leq 0\}$.*

Proof. (1)⇒(2) For $v \in \mathrm{Dom}A$,

$$0 = \partial_t(\mathrm{e}^{-\mathrm{i}tA}v|\mathrm{e}^{-\mathrm{i}tA}v)\Big|_{t=0} = -\mathrm{i}(Av|v) + \mathrm{i}(v|Av).$$

(2)⇒(1) We know that $\mathrm{e}^{-\mathrm{i}tA}$ is the generator of a strongly continuous contractive semigroup. For $v \in \mathrm{Dom}A$,

$$0 = \partial_t(\mathrm{e}^{-\mathrm{i}tA}v|\mathrm{e}^{-\mathrm{i}tA}v)$$

Hence, for $v \in \mathrm{Dom}A$, $\|\mathrm{e}^{-\mathrm{i}tA}v\|^2 = \|v\|^2$. □

Theorem 3.14 *Let A be hermitian. Then the following conditions are equivalent:*

(1) $\mathrm{sp}A \subset \{\mathrm{Im} z \leq 0\}$.

(2) *There exists z_0 with $\mathrm{Im} z_0 > 0$ and $\mathrm{Ran}\,(z_0 - A) = \mathcal{V}$.*

(3) *A is closed and there esxists z_0 with $\mathrm{Im} z_0 > 0$ and $\mathrm{Ran}\,(z_0 - A)$ dense in $\mathcal{V}$.*

(4) *A is closed and there exists z_0 with $\mathrm{Im} z_0 > 0$ and $\mathrm{Ker}(\overline{z}_0 - A^*) = \{0\}$.*

3.7 Self-adjoint operators

Let T be a densely defined operator on $\mathcal{V}$. T is self-adjoint iff $T^* = T$, that means if for $w \in \mathcal{W}$ there exists $y \in \mathcal{V}$ such that

$$(y|v) = (w|Tv), \ v \in \mathrm{Dom}T,$$

then $w \in \mathrm{Dom}T$ and $Tw = y$.

Theorem 3.15 *Every self-adjoint operator is hermitian and closed. If A is bounded, then it is self-adjoint iff it is hermitian.*

Theorem 3.16 *Fix $z_\pm$ with $\pm\mathrm{Im}z_\pm > 0$. Let A be hermitian. Then the following conditions are necessary and sufficient for A to be self-adjoint:*

(1) $\mathrm{sp}A \subset \mathbb{R}$.

(2) $z_\pm \not\in \mathrm{sp}A$.

(3) $\mathrm{Ran}\,(z_\pm - A) = \mathcal{V}$.

(4) *A is closed and $\mathrm{Ran}\,(z_\pm - A)$ is dense in $\mathcal{V}$.*

(5) *A is closed and $\mathrm{Ker}(\overline{z}_\pm - A^*) = \{0\}$.*

Theorem 3.17 *Let $\lambda_0 \in \mathbb{R}$. Let A be hermitian. Then the following conditions are sufficient for A to be self-adjoint:*

(1) $\lambda_0 \not\in \mathrm{sp}A$.

(2) $\mathrm{Ran}\,(\lambda_0 - A) = \mathcal{V}$.

(3) *A is closed and $\mathrm{Ran}\,(\lambda_0 - A)$ is dense in $\mathcal{V}$.*

(4) *A is closed and $\mathrm{Ker}(\lambda_0 - A^*) = \{0\}$.*

Theorem 3.18 *Let A be self-adjoint. Then $U := (A + \mathrm{i})(A - \mathrm{i})^{-1}$ is a unitary operator with $\mathrm{sp}U = (\mathrm{sp}^{\mathrm{ext}}A + \mathrm{i})(\mathrm{sp}^{\mathrm{ext}}A - \mathrm{i})^{-1}$.*

If $f \in C(\mathrm{sp}^{\mathrm{ext}}A)$, we can define

$$f(A) := f(\mathrm{i}(U + \mathrm{i})(U - 1)^{-1}).$$

Of course, we can also apply the functional calculus for measurable functions. In particular, the function $\mathrm{sp}A \ni x \mapsto \mathrm{id}(x) := x$ is a measurable function on $\mathrm{sp}A$. We have $\mathrm{id}A = A$.

Theorem 3.19 (Stone Theorem) *Let A be an operator. Then the following conditions are equivalent:*

(1) *$\mathrm{i}A$ is the generator of a strongly continuous group of unitary operators.*

(2) *A is self-adjoint.*

Proof. To prove (1)⇒(2), suppose that $\mathbb{R} \mapsto U(t)$ is a strongly continuous unitary group. Let $-\mathrm{i}A$ be its generator. Then $[0, \infty[\ni U(t), U(-t)$ are semigroups of contractions with the generators $\mathrm{i}A$ and $-\mathrm{i}A$. By Theorem 3.19, A is hermitian and $\mathrm{sp}A \subset \mathbb{R}$. Hence A is self-adjoint.

(2)⇒(1) follows by the spectral theorem. □

3.8 Essentially self-adjoint operators

Definition 3.20 *An operator* $A : \mathcal{V} \to \mathcal{V}$ *is essentially self-adjoint iff* A^{cl} *is self-adjoint.*

Theorem 3.21 *Every essentially self-adjoint operator is hermitian and closable.*

Theorem 3.22 *Fix* $z_\pm$ *with* $\pm \mathrm{Im} z_\pm > 0$ *Let* A *be hermitian. Then the following conditions are necessary and sufficient for* A *to be essentially self-adjoint:*

(1) A^* *is self-adjoint.*

(2) $\mathrm{Ran}\,(z_\pm - A)$ *is dense in* $\mathcal{V}$.

(3) $\mathrm{Ker}(\overline{z}_\pm - A^*) = \{0\}$.

Theorem 3.23 *Let* $\lambda_0 \in \mathbb{R}$. *Let* A *be hermitian. Then the following conditions are sufficient for* A *to be self-adjoint:*

(1) $\mathrm{Ran}\,(\lambda_0 - A)$ *is dense in* $\mathcal{V}$.

(2) $\mathrm{Ker}(\lambda_0 - A^*) = \{0\}$.

3.9 Scale of Hilbert spaces

Let B be a positive operator on $\mathcal{V}$ with $B \geq 1$. We define the family of Hilbert spaces $\mathcal{V}_\alpha$, $\alpha \in \mathbb{R}$ as follows. For $\alpha \geq 0$, we set $\mathcal{V}_\alpha := \mathrm{Ran}\, B^{-\alpha/2}$ with the scalar product

$$(v|w)_\alpha := (v|B^\alpha w),$$

and $\mathcal{V}_{-\alpha} := \mathcal{V}_\alpha^*$, (where $\mathcal{V}_\alpha^*$ denotes the space of bounded antilinear functionals on $\mathcal{V}_\alpha$). Note that we have the identification $\mathcal{V} = \mathcal{V}^*$, hence both definitions give $\mathcal{V}_0 = \mathcal{V}$.

It is clear, that for $0 \leq \alpha \leq \beta$, $\mathcal{V} \supset \mathcal{V}_\alpha \supset \mathcal{V}_\beta$. Hence $\mathcal{V}_{-\alpha} = \mathcal{V}_\alpha^*$ can be identified with a subspace of $\mathcal{V}_{-\beta} = \mathcal{V}_\beta^*$. Thus we obtain $\mathcal{V}_\alpha \supset \mathcal{V}_\beta$ for any $\alpha \leq \beta$.

Note that for $\alpha > 0$ $\mathcal{V}$ is embedded in $\mathcal{V}_{-\alpha}$ and for $v, w \in \mathcal{V}$

$$(v|w)_{-\alpha} = \left(B^{-\alpha/2} v | B^{-\alpha/2} w \right).$$

Moreover, $\mathcal{V}$ is dense in $\mathcal{V}_{-\alpha}$.

It is easy to see that, for $\alpha \geq 0$, $B_0^{-\alpha} := B^{-\alpha}$ is a unitary operator from $\mathcal{V}_0$ to $\mathcal{V}_{2\alpha}$. Moreover, the operator B^α, defined by the functional calculus (or as the inverse of the bounded operator $B^{-\alpha}$, with the domain $\mathcal{V}_{2\alpha}$ extends to a unitary operator from $\mathcal{V}_0$ to $\mathcal{V}_{-\alpha}$, which we will denote by B_0^α. For any α, β, setting $B_\alpha^{\alpha-\beta} := B_0^{-\beta}(B_0^{-\alpha})^{-1}$ we obtain a unitary operator from $\mathcal{V}_{2\alpha}$ to $\mathcal{V}_{2\beta}$. These operators satisfy the chain rule:

$$B_{\alpha+2\beta}^{-\gamma} B_\alpha^{-\beta} = B_\alpha^{-\gamma-\beta}.$$

Sometimes we will use a different notation: $B^{-\alpha}\mathcal{V} = \mathcal{V}_{2\alpha}$. If A is a self-adjoint operator, then we will use the notation $\langle A \rangle := (1 + A^2)^{1/2}$.

Lemma 3.24 *Let* $0 \leq \alpha \leq 1$. *Then* $\mathrm{Dom} B = \{v \in \mathcal{V}_\alpha \ : \ B_\alpha v \in \mathcal{V}\}$.

3.10 Relative operator boundedness

Theorem 3.25 *Let* A *be a closed operator and* B *an operator with* $\mathrm{Dom} B \supset \mathrm{Dom} A$. *Then the following statements are equivalent:*

(1) *B has the A-bound equal to* a_1, *that is*

$$\inf_{\nu>0} \sup_{v\neq 0,\ v\in \mathrm{Dom} A} \left(\frac{\|Bv\|^2}{\|Av\|^2 + \nu^2 \|v\|} \right)^{\frac{1}{2}} = a_1 .$$

(2)

$$\inf_{\nu>0} \|B(A^*A+\nu^2)^{-1/2}\| = a_1 .$$

If, moreover, A is self-adjoint, then the above statements are equivalent to

(3) $\inf\limits_{\nu>0} \|B(\mathrm{i}\nu - A)^{-1}\| = a_1$

Proof. The equivalence of (1) and (2) is evident.

The equivalence of (2) and (3) for a self-adjoint A is the consequence of the unitarity of

$$(A^2+\nu^2)^{-1/2}(\mathrm{i}\nu - A).$$

□

Theorem 3.26 (Kato-Rellich) *Let A be self-adjoint, B hermitian. Let B be A-bounded with the* $A-$*bound* < 1. *Then*

(1) $A+B$ *is self-adjoint on* $\mathrm{Dom}A$.

(2) *If A is essentally self-adjoint on* $\mathcal{D}$, *then* $A+B$ *is essentially self-adjoint on* $\mathcal{D}$.

Proof. Clearly, $A+B$ is hermitian on $\mathrm{Dom}A$. Moreover, for some ν, $\|B(\mathrm{i}\nu - A)^{-1}\| < 1$ and (which is equivalent by the unitarity of $(A-\mathrm{i}\nu)(A+\mathrm{i}\nu)^{-1}$), $\|B(-i\nu - A)^{-1}\| < 1$. Hence, $\mathrm{i}\nu - A - B$ and $-\mathrm{i}\nu - A - B$ are invertible. □

Let us note an improved version of the notion of the operator boundedness:

Theorem 3.27 *Let A be a closed operator and B an operator with* $\mathrm{Dom}B \supset \mathrm{Dom}A$. *Then the following statements are equivalent:*

(1)

$$a_2 = \inf_{\mu,\nu>0} \sup_{v\neq 0} \left(\frac{\|Bv\|^2}{\|(A-\mu)v\|^2 + \nu^2 \|v\|} \right)^{\frac{1}{2}} .$$

(2)

$$\inf_{\mu,\nu>0} \|B((A-\mu)^*(A-\mu)+\nu^2)^{-1/2}\| = a_2 .$$

If, moreover, A is self-adjoint, then the above statements are equivalent to

(3) $\inf\limits_{\mu,\nu>0} \|B(\mu + \mathrm{i}\nu - A)^{-1}\| = a_2$

Note hat the analog of Theorem 3.26 is true with a_1 replaced with a_2.

3.11 Relative form boundedness

Theorem 3.28 *Let A be a self-adjoint operator. Let B be a bounded operator from $(1+|A|)^{-1/2}\mathcal{H}$ to $(1+|A|)^{1/2}\mathcal{H}$. Then the following statements are equivalent:*
(1)

$$\inf_{\mu,\nu}\|(A-\mu)^2+\nu^2)^{-\frac14}B((A-\mu)^2+\nu^2)^{-\frac14}\| = a_3.$$

(2)

$$\inf_{\mu,\nu>0}\|(\mu+\mathrm{i}\nu-A)^{-\frac12}v,(\mu+\mathrm{i}\nu-A)^{-\frac12}\| = a_3.$$

If the conditions of the above theorem are satified, then we say that the A-form-bound of B equals a_3.

Theorem 3.29 *Let A be a self-adjoint operator. Let B have the A-form-bound less than* 1*. Then there exists a open subsets in the upper and lower complex half-plane such that the series*

$$R(z) := \sum_{j=0}^{\infty}(z-A)^{-1}(B(z-A)^{-1})^j$$

is convergent. Moreover, $R(z)$ is a resolvent of a self-adjoint operator, which will be called the form sum of A and B. If A is bounded from below, then so is $A+B$ and $\mathrm{Dom}|A+B|^{\frac12} = \mathrm{Dom}|A|^{\frac12}$.

3.12 Non-maximal operators

Theorem 3.30 $\dim\mathrm{Ran}\,(z-A)^{\perp} = \dim\mathrm{Ker}(\overline{z}-A^*)$ *is a constant function on connected components of* $\mathbb{C}\backslash(\mathrm{Num}A)^{\mathrm{cl}}$.

Proof. Let us show that if $|\alpha|<\lambda$, then

$$\mathrm{Ran}\,(\mathrm{i}\lambda-A)\cap\mathrm{Ran}\,(\mathrm{i}\lambda+\alpha-A)^{\perp} = \{0\}. \tag{3.14}$$

Let $w\in\mathrm{Ran}\,(\mathrm{i}\lambda-A)$. Then there exists $v\in\mathrm{Dom}A$ such that

$$w = (\mathrm{i}\lambda-A)v$$

and $\|v\|\leq\lambda^{-1}\|w\|$. If moreover, $w\in\mathrm{Ran}\,(\mathrm{i}\lambda+\alpha-A)^{\perp} = \mathrm{Ker}(-\mathrm{i}\lambda-\overline{\alpha}-A^*)$, then

$$\begin{aligned}0 &= ((-\mathrm{i}\lambda+\overline{\alpha}-A^*)w|v)\\ &= (w|(\mathrm{i}\lambda-A)v)+\alpha(w|v)\\ &= \|w\|^2+\alpha(w|v).\end{aligned}$$

But

$$|\|w\|^2+\alpha(w|v)| \geq (1-|\alpha|/|\lambda|)\|w\|^2 > 0,$$

which is a contradiction and completes the proof of (3.14).

Now (3.14) implies that $\dim\mathrm{Ran}\,(\mathrm{i}\lambda-A)\leq\dim\mathrm{Ran}\,(\mathrm{i}\lambda+\alpha-A)$. □

3.13 Hermitian operators II

Let A be closed hermitian.

Theorem 3.31 *The so-called defect indices*

$$n_\pm := \dim \mathrm{Ker}(z - A^*),\ \ z \in \mathbb{C}_\pm$$

do not depend on λ. *One of the following possibilities is true:*

$$\begin{aligned}
&1)\ \mathrm{sp}A \subset \mathbb{R}, && n_\pm = 0,\ \ A \textit{ is self-adjoint;}\\
&2)\ \mathrm{sp}A = \{\mathrm{Im}z \geq 0\}, && n_+ \neq 0, n_- = 0,\ \ A \textit{ is not self-adjoint};\\
&3)\ \mathrm{sp}A = \{\mathrm{Im}z \leq 0\}, && n_+ = 0, n_- \neq 0,\ \ A \textit{ is not self-adjoint};\\
&4)\ \mathrm{sp}A = \mathbb{C} && n_+ \neq 0, n_- \neq 0,\ \ A \textit{ is not self-adjoint.}
\end{aligned}$$

Proof. This is a special case of Theorem 3.30. □

Definition 3.32 *Let A be hermitian and closed. Define on* $\mathrm{Dom}A^*$ *the following scalar product:*

$$(v|w)_{A^*} := (v|w) + (A^*v|A^*w)$$

and the following antihermitian form:

$$[v|w]_{A^*} := (A^*v|w) - (v|A^*w).$$

The A^*-completeness and the A^*-orthogonality is defined using the scalar product $(\cdot|\cdot)_{A^*}$. A space is A^*-hermitian iff $[\cdot|\cdot]_{A^*}$ vanishes on this subspace.

Theorem 3.33 (1) *Every closed extension of A is a restriction of A^* to an A^*-closed subspace in* $\mathrm{Dom}A^*$ *containing* $\mathrm{Dom}A$.

(2)

$$\mathrm{Dom}A^* = \mathrm{Dom}A \oplus \mathrm{Ker}(A^* + \mathrm{i}) \oplus \mathrm{Ker}(A^* - \mathrm{i})$$

and the components in the above direct sum are A^*-closed, A^*-orthogonal and

$$(w \oplus w_+ \oplus w_-|v \oplus v_+ \oplus v_-)_{A^*} = (w|v) + (Aw|Av) + 2(w_+|v_+) + 2(w_-|v_-),$$

$$[w \oplus w_+ \oplus w_-|v \oplus v_+ \oplus v_-]_{A^*} = 2i(w_+|v_+) - 2i(w_-|v_-).$$

Proof. (1) is obvious. In (2) the A^*-orthogonality and the A^*-closedness are easy.

Let $w \in \mathrm{Dom}A^*$ and

$$w \perp \mathrm{Dom}A \oplus \mathrm{Ker}(A^* + \mathrm{i}) \oplus \mathrm{Ker}(A^* - \mathrm{i})$$

in the sense of the product $(\cdot|\cdot)_{A^*}$. In particular, for $v \in \mathrm{Dom}A$ we have

$$0 = (A^*w|A^*v) + (w|v) = (A^*w|Av) + (w|v).$$

Hence $A^*w \in \mathrm{Dom}A^*$ and

$$A^*A^*w = -w.$$

Therefore,

$$(A^* + \mathrm{i})(A^* - \mathrm{i})w = 0.$$

Thus

$$(A^* - \mathrm{i})w \in \mathrm{Ker}(A^* + \mathrm{i}). \tag{3.15}$$

If $y \in \mathrm{Ker}(A^* + \mathrm{i})$, then

$$\mathrm{i}(y|(A^* - \mathrm{i})w) = (A^*\eta|A^*w) + (\eta|w) = (\eta|w)_{A^*} = 0$$

In particular, setting $y = (A^* - \mathrm{i})w$ and using (3.15), we get

$$w \in \mathrm{Ker}(A^* - \mathrm{i}).$$

But $w \perp \mathrm{Ker}(A^* - \mathrm{i})$, hence $w = 0$. □

Every A^*-closed subspace containing $\mathrm{Dom}A$ is of the form $\mathrm{Dom}A \oplus \mathcal{Z}$, where

$$\mathcal{Z} \subset \mathrm{Ker}(A^* + \mathrm{i}) \oplus \mathrm{Ker}(A^* - \mathrm{i}).$$

If

$$A \subset B \subset A^*,$$

then the subspace $\mathcal{Z}$ corresponding to B will be denoted by $\mathcal{Z}_B$. We will write

$$\mathcal{Z}^{\mathrm{per}} := \{v \in \mathrm{Ker}(A^* - \mathrm{i}) \oplus \mathrm{Ker}(A^* - \mathrm{i}) \ : \ [z, v]_{A^*} = 0, \ z \in \mathcal{Z}\}.$$

The subspace $\mathcal{Z}$ is A^*-hermitian iff

$$\mathcal{Z}^{\mathrm{per}} \supset \mathcal{Z}.$$

Theorem 3.34 *We have*

$$\mathcal{Z}_{B^*} = (\mathcal{Z}_B)^{\mathrm{per}}.$$

In particular, B is hermitian iff $\mathcal{Z}$ is A^*-hermitian. Every A^*-hermitian subspace corresponds to a partial isometry $U : \mathrm{Ker}(A^* + \mathrm{i}) \to \mathrm{Ker}(A^* - \mathrm{i})$. Then

$$\mathcal{Z} := \{z \oplus Uz \ : \ z \in \mathrm{Ran}\, U^*U\}.$$

B is self-adjoint iff U is unitary.

4 Sesquilinear forms

4.1 Sesquilinear forms

Let $\mathcal{V}, \mathcal{W}$ be complex spaces. We say that $\mathfrak{t}$ is a sesquilinear quasiform on $\mathcal{W} \times \mathcal{V}$ iff there exist subspaces $\mathrm{Dom}_{\mathrm{l}}\mathfrak{t} \subset \mathcal{W}$ and $\mathrm{Dom}_{\mathrm{r}}\mathfrak{t} \subset \mathcal{V}$ such that

$$\mathrm{Dom}_{\mathrm{l}}\mathfrak{t} \times \mathrm{Dom}_{\mathrm{r}}\mathfrak{t} \ni (w, v) \mapsto \mathfrak{t}(w, v) \in \mathbb{C}$$

is a sesquilinear map. From now on by a sesquilinear form we will mean a sesquilinear quasiform.

We define a form $\mathfrak{t}^*$ with the domains $\mathrm{Dom}_{\mathrm{l}}\mathfrak{t}^* := \mathrm{Dom}_{\mathrm{r}}\mathfrak{t}$, $\mathrm{Dom}_{\mathrm{r}}\mathfrak{t}^* := \mathrm{Dom}_{\mathrm{l}}\mathfrak{t}$, by the formula $\mathfrak{t}^*(v, w) := \overline{\mathfrak{t}(w, v)}$. If $\mathfrak{t}_1$ are $\mathfrak{t}_2$ forms, then we define $\mathfrak{t}_1 + \mathfrak{t}_2$ with the domain $\mathrm{Dom}_{\mathrm{l}}\mathfrak{t}_1 + \mathfrak{t}_2 := \mathrm{Dom}_{\mathrm{l}}\mathfrak{t}_1 \cap \mathrm{Dom}_{\mathrm{l}}\mathfrak{t}_1$, $\mathrm{Dom}_{\mathrm{r}}\mathfrak{t}_1 + \mathfrak{t}_2 := \mathrm{Dom}_{\mathrm{r}}\mathfrak{t}_1 \cap \mathrm{Dom}_{\mathrm{r}}\mathfrak{t}_1$ by $(\mathfrak{t}_1 + \mathfrak{t}_2)(w, v) := \mathfrak{t}_1(w, v) + \mathfrak{t}_2(w, v)$. We write $\mathfrak{t}_1 \subset \mathfrak{t}_2$ if $\mathrm{Dom}_{\mathrm{l}}\mathfrak{t}_1 \subset \mathrm{Dom}_{\mathrm{l}}\mathfrak{t}_2$, $\mathrm{Dom}_{\mathrm{r}}\mathfrak{t}_1 \subset \mathrm{Dom}_{\mathrm{r}}\mathfrak{t}_2$, and $\mathfrak{t}_1(w, v) = \mathfrak{t}_2(w, v)$ $w \in \mathrm{Dom}_{\mathrm{l}}\mathfrak{t}_1$, $v \in \mathrm{Dom}_{\mathrm{r}}\mathfrak{t}_1$.

$\mathfrak{t}$ is bounded iff

$$|\mathfrak{t}(w, v)| \leq c\|w\|\|v\|, \quad w \in \mathrm{Dom}_{\mathrm{l}}\mathfrak{t}, \quad v \in \mathrm{Dom}_{\mathrm{r}}\mathfrak{t}.$$

From now on, we will usually assume that $\mathcal{W} = \mathcal{V}$ and $\mathrm{Dom}_{\mathrm{l}}\mathfrak{t} = \mathrm{Dom}_{\mathrm{r}}\mathfrak{t}$ and the latter subspace will be simply denoted by $\mathrm{Dom}\mathfrak{t}$.

Recall that the numerical range of the form $\mathfrak{t}$ is defined as

$$\mathrm{Num}\mathfrak{t} := \{\mathfrak{t}(v) \ : \ v \in \mathrm{Dom}\mathfrak{t}, \ \|v\| = 1\}.$$

We proved that $\mathrm{Num}\mathfrak{t}$ is a convex set.

The form $\mathfrak{t}$ is bounded iff $\mathrm{Num}\mathfrak{t}$ is bounded. Equivalently, $|\mathfrak{t}(v)| \leq c\|v\|^2$.

$\mathfrak{t}$ is bounded from below, if there exists c such that

$$\mathrm{Num}\mathfrak{t} \subset \{z \ : \ \mathrm{Re} z > c\}.$$

$\mathfrak{t}$ is hermitian iff $\mathrm{Num}\mathfrak{t} \subset \mathbb{R}$. The equivalent condition: $\mathfrak{t}(w, v) = \overline{\mathfrak{t}(v, w)}$.

If T is an operator on $\mathcal{V}$, then $\mathfrak{t}(w, v) := (w, Tv)$ with the domain $\mathrm{Dom}T$ is a form called the form associated with the operator T. Clearly, $\mathrm{Num}\mathfrak{t} = \mathrm{Num}T$.

4.2 Closed positive forms

Let $\mathfrak{s}$ be a positive form.

Definition 4.1 *We say that $\mathfrak{s}$ is a closed form iff* $\mathrm{Dom}\mathfrak{s}$ *with the scalar product*

$$(w, v)_{\mathfrak{s}} := (\mathfrak{s} + 1)(w, v), \quad w, v \in \mathrm{Dom}\mathfrak{s},$$

is a Hilbert space.

Theorem 4.2 *The form $\mathfrak{s}$ is closed iff for any sequence (x_n) in* $\mathrm{Dom}\mathfrak{s}$*, if $x_n \to x$ and $\mathfrak{s}(x_n - x_m) \to 0$, then $x \in$* $\mathrm{Dom}\mathfrak{s}$ *and $\mathfrak{s}(x_n - x) \to 0$.*

Example 4.3 *Let A be an operator. Then $(Aw|Av)$ with the domain* $\mathrm{Dom}A$ *is a closed form iff A is closed.*

4.3 Closable positive forms

Let $\mathfrak{s}$ be a positive form.

Definition 4.4 *We say that $\mathfrak{s}$ is a closable form iff there exists a closed form $\mathfrak{s}_1$ such that $\mathfrak{s} \subset \mathfrak{s}_1$.*

Theorem 4.5 (1) *The form $\mathfrak{t}$ is closable $\Leftrightarrow$ for any sequence $(x_n) \subset$* $\mathrm{Dom}\mathfrak{s}$*, if $x_n \to 0$ and $\mathfrak{s}(x_n - x_m) \to 0$, then $\mathfrak{s}(x_n) \to 0$.*

(2) *If $\mathfrak{s}$ is closable, then there exists the smallest closed form $\mathfrak{s}_1$ such that $\mathfrak{s} \subset \mathfrak{s}_1$. We will denote it by $\mathfrak{s}^{\mathrm{cl}}$.*

(3) $\mathrm{Num}\mathfrak{s}$ *is dense in* $\mathrm{Num}\mathfrak{s}^{\mathrm{cl}}$

Proof. (1) $\Rightarrow$ follows immediately from Theorem 4.2.

To prove (1) $\Leftarrow$, define $\mathfrak{s}_1$ as follows: $v \in \mathrm{Dom}\mathfrak{s}_1$, iff there exists a sequence $(v_n) \subset \mathrm{Dom}\mathfrak{s}$ such that $v_n \to v$ and $\mathfrak{s}(v_n - v_m) \to 0$. We set then $\mathfrak{s}_1(v) := \lim_{n\to\infty} \mathfrak{s}(v_n)$ (the limit exists, because $\mathfrak{s}(v_n)$ is a Cauchy sequence).

To show that the definition is correct, suppose that $w_n \in \mathrm{Dom}\mathfrak{s}$, $w_n \to v$ and $\mathfrak{s}(w_n - w_m) \to 0$. Then $v_n - w_n - (v_m - w_m) \to 0$ and $v_n - w_n \to 0$. By the hypothesis we get $\mathfrak{s}(v_n - w_n) \to 0$. Hence, $\lim_{n\to\infty} \mathfrak{s}(v_n) = \lim_{n\to\infty} \mathfrak{s}(w_n)$. Thus the definition of $\mathfrak{s}_1$ does not depend on the choice of the sequence v_n. It is clear that $\mathfrak{s}_1$ is a closed form containing $\mathfrak{s}$. Hence $\mathfrak{s}$ is closable.

To prove (2) note that the form $\mathfrak{s}_1$ constructed above is the smallest closed form containg $\mathfrak{s}$. $\square$

Example 4.6 *Let A be an operator. Then $(Aw|Av)$ with the domain* $\mathrm{Dom}A$ *is closable iff A is a closable operator. Then $(A^{\mathrm{cl}}w|A^{\mathrm{cl}}v)$ with the domain* $\mathrm{Dom}A^{\mathrm{cl}}$ *is its closure.*

4.4 Operators associated with positive forms

Let S be a positive self-adjoint operator. We define the form $\mathfrak{s}$ as follows: $\mathrm{Dom}\mathfrak{s} := \mathrm{Dom}S^{1/2}$ and $\mathfrak{s}(v,w) := (S^{1/2}v|S^{1/2}w)$. We will say that $\mathfrak{s}$ is the form associated with S.

Theorem 4.7 (1) $\mathfrak{s}$ *is a closed form.*

(2) $\mathrm{Dom}S$ *is an essential domain for* $\mathfrak{s}$*.*

(3) $\mathrm{Num}S$ *is dense in* $\mathrm{Num}\mathfrak{s}$*.*

The next theorem describes the converse construction.

Theorem 4.8 *Let* $\mathfrak{s}$ *be a densely defined closed positive form. Then there exists a unique positive self-adjoint operator* S *such that* $\mathrm{Dom}\mathfrak{s} = \mathrm{Dom}S^{1/2}$ *and* $\mathfrak{s}(v,w) := (S^{1/2}v|S^{1/2}w)$*. We will say that* S *is the operator associated with the form* $\mathfrak{s}$*.*

Proof. For $w \in \mathcal{V}$, $v \in \mathrm{Dom}\mathfrak{s}$, we have

$$|(v|w)| \leq \|v\|\|w\| \leq \|v\|_{\mathfrak{s}}\|w\|.$$

By the Riesz lemma, there exists $A : \mathcal{V} \to \mathrm{Dom}\mathfrak{s}$ such that

$$(v|w) = (v|Aw)_{\mathfrak{s}}, \tag{4.16}$$

$$\|Aw\| \leq \|Aw\|_{\mathfrak{s}} \leq \|w\|.$$

$\mathrm{Ker}A = \{0\}$, because $Aw = 0$ implies $(v|w) = 0$ for $v \in \mathrm{Dom}\mathfrak{s}$, and $\mathrm{Dom}\mathfrak{s}$ s dense in $\mathcal{V}$. Besides, A is self-adjoint. Putting

$$S := A^{-1} - 1$$

we define a positive self-adjoint operator.

$$\mathfrak{s}(v,y) = (v|Sy), \quad v \in \mathrm{Dom}\mathfrak{s}, \quad y \in \mathrm{Dom}S = \mathrm{Ran}\,A.$$

Let us show that $\mathrm{Dom}\mathfrak{s}$ is an essential domain for $\mathfrak{s}$. Let $v \in \mathrm{Dom}\mathfrak{s}$ is $\mathfrak{s}$-orthogonal to $\mathrm{Ran}\,A = \mathrm{Dom}S$. Then v is orthogonal to $\mathrm{Dom}\mathfrak{s}$—see (4.16). Hence $v = 0$.

Define $\mathfrak{s}_1$ by $\mathrm{Dom}\mathfrak{s}_1 = \mathrm{Dom}S^{\frac{1}{2}}$ and $\mathfrak{s}_1(w,v) = (S^{\frac{1}{2}}w|S^{1/2}v)$. The form $\mathfrak{s}$ and $\mathfrak{s}_1$ coincide on $\mathrm{Dom}S \subset \mathrm{Dom}S^{\frac{1}{2}} \cap \mathrm{Dom}\mathfrak{s}$. We proved above that $\mathrm{Dom}S$ is an essential domain for $\mathfrak{s}$. The form $\mathfrak{s}_1$ is obviously closed and $\mathrm{Dom}\mathfrak{s}_1$ is an essential domain for $\mathfrak{s}_1$. Hence, $\mathfrak{s}_1 = \mathfrak{s}$. □

4.5 Polar decomposition

Theorem 4.9 *Let* A *be a densely defined closed operator. Let* B *be the operator associated to the form* $(Aw|Av)$*. Let* $\mathcal{V}_\alpha$ *be the scale of spaces* $(B+1)^{-\alpha}\mathcal{V}$*, so that* $\mathcal{V}_1 = \mathrm{Dom}A$ *and* $\mathcal{V}_{-1} = \mathcal{V}_1^*$*. Then*

(1) A*, treated as an operator from* $\mathcal{V}_1$ *to* $\mathcal{V}_0$*, and denoted* A_1*, is bounded.*

(2) A^* *extends by density to a bounded operator, denoted* A_0^* *from* $\mathcal{V}_0$ *to* $\mathcal{V}_{-1}$*. We have* $A_0^* = (A_1)^*$*.*

(3) $\mathrm{Dom}A^* = \{v \in \mathcal{V} \ :\ A_0^*v \in \mathcal{V}\}$*.*

(4) $\mathrm{Dom}B = \{v \in \mathrm{Dom}A \ :\ Av \in \mathrm{Dom}A^*\}$ *and for* $v \in \mathrm{Dom}B$*,* $Bv = A^*Av$*.*

Proof. (1) is obvious.

To see (2), note that for $v \in \mathrm{Dom}A^*$, $w \in \mathrm{Dom}A$ we have

$$|(A^*v|w)| = (v|Aw)| \leq \|v\|\|w\|_1.$$

Hence $\|A^*v\|_{-1} \leq \|v\|$, and so $A^* : \mathcal{V} \to \mathcal{V}_{-1}$ is bounded. $\mathrm{Dom}A^*$ is dense in $\mathcal{V}$. Hence A^* extends to a bounded operator $A_0^* : \mathcal{V} \to \mathcal{V}_{-1}$.

To prove (3), let $v \in \mathcal{V}$, $A_0^* v = w \in \mathcal{V}$. Then there exists $(v_n) \subset \mathrm{Dom}A^*$ such that $v_n \to v$ in the norm of $\mathcal{V}$ and $A^*v_n \to w$ in the norm of $\mathcal{V}_{-1}$. Hence for $x \in \mathrm{Dom}A$,

$$(v_n|Ax) = (A^*v_n|x) \to (w|x).$$

Therefore, $(w|x) = (v|Ax)$. Hence, $v \in \mathrm{Dom}A^*$ and $A^*v = w$.

To prove (4), denote B_1 as the extension of B to the operator from $\mathcal{V}_1$ to $\mathcal{V}_{-1}$. Note that B_1 is bounded.

We have $B_1 = A_0^* A_1$. In fact, for $v, w \in \mathcal{V}_1$

$$(w|B_1 v) = (B^{1/2}w|B^{1/2}v) = (Aw|Av) = (w|A_0^* A_1 v).$$

Now

$$\begin{aligned} \mathrm{Dom}B &= \{v \in \mathcal{V}_1 \ : \ B_1 v \in \mathcal{V}\} \\ &= \{v \in \mathcal{V}_1 \ : \ A_0^* A_1 v \in \mathcal{V}\} \\ &= \{v_1 \ : \ Av \in \mathrm{Dom}A^*\}. \end{aligned}$$

□

Motivated by the above theorem we will write A^*A for B.

Theorem 4.10 *Let A be closed. Then there exist a unique positive operator $|A|$ and a unique partial isometry U such that* $\mathrm{Ker}U = \mathrm{Ker}A$ *and* $A = U|A|$. *We have then* $\mathrm{Ran}\, U = \mathrm{Ran}\, A^{\mathrm{cl}}$.

Proof. The operator A^*A is positive. By the spectral theorem, we can then define

$$|A| := \sqrt{A^*A}.$$

On $\mathrm{Ran}\,|A|$ the operator U is defined by

$$U\ |A|v := Av.$$

It is isometric, because

$$\||A|v\|^2 = (v||A|^2 v) = (v|A^*Av) = \|Av\|^2,$$

and correctly defined. We can extend it to $(\mathrm{Ran}\,|A|)^{\mathrm{cl}}$ by continuity. On $\mathrm{Ker}|A| = (\mathrm{Ran}\,|A|)^{\mathrm{cl}}$, we extend it by putting $Uv = 0$. □

4.6 Sectorial forms

A subset of $\mathbb{C}$ of the form $\mathrm{Sec}(a,\theta) := a + \{z \ : \ |\arg z| < \theta\}$ with $\theta < \frac{\pi}{2}$ will be called a sector. a will be called its tip and θ its angle. We say that a form $\mathfrak{t}$ is sectorial iff there exists $a \in \mathbb{C}$ and $\theta < \frac{\pi}{2}$ such that $\mathrm{Num}\mathfrak{t} \subset \mathrm{Sec}(a,\theta)$.

Lemma 4.11 *Let $\mathfrak{t}$ be a sectorial form with the sector given by a, θ. Then*

$$|(\mathfrak{t}-a)(w,v)| \leq (1+\tan\theta)\mathrm{Re}(\mathfrak{t}-a)(w)^{\frac{1}{2}}\mathrm{Re}(\mathfrak{t}-a)(v)^{\frac{1}{2}}.$$

Clearly, if $\mathfrak{t}$ is sectorial and a is the tip of a sector containing $\mathrm{Num}\mathfrak{t}$, then $\mathrm{Re}\mathfrak{t}+\mathrm{Re}a$ is a positive form. We say that a sectorial form $\mathfrak{t}$ is closed iff $\mathrm{Re}\mathfrak{t}$ is closed. We say that a sectorial form $\mathfrak{t}$ is closable iff $\mathrm{Re}\mathfrak{t}$ is closable.

It is easy to see, using Lemma 4.11, that Theorems 4.2 and 4.5 remain true if the form $\mathfrak{s}$ is assumed to be sectorial.

4.7 Operators associated with sectorial forms

Definition 4.12 *An operator T is called sectorial iff the associated form is sectorial, that means, if* $\mathrm{Num}T \subset \mathrm{Sec}(a,\theta)$. *A maximal sectorial operator is called shortly m-sectorial.*

Theorem 4.13 (1) *Let $\mathfrak{t}$ be a sectorial form. Then there exists a unique m-sectorial operator T such that* $\mathrm{Dom}T \subset \mathrm{Dom}\mathfrak{t}$ *and*

$$\mathfrak{t}(w,v) = (w|Tv), \quad v \in \mathrm{Dom}T, \ w \in \mathrm{Dom}\mathfrak{t}.$$

T is called the operator associated with the form $\mathfrak{t}$ and denoted by $T_{\mathfrak{t}}$.

(2) $\mathrm{Dom}T$ *is an essential domain for* $\mathfrak{t}$.

(3) $\mathrm{Num}T$ *is dense in* $\mathrm{Num}\mathfrak{t}$.

(4) *If $\mathfrak{t}$ is bounded, then so is T.*

We will assume that the sector of $\mathfrak{t}$ has the tip at 0. We will write $\mathfrak{s} := \mathrm{Re}\mathfrak{t}$.

Lemma 4.14 *There exists an invertible operator* $B \in B(\mathrm{Dom}\mathfrak{s})$ *such hat*

$$(\mathfrak{t}+1)(w,v) = \mathfrak{s}(w,Bv) + (w|Bv).$$

Proof. By Lemma 4.11, the form $\mathfrak{t}+1$ is bounded in the Hilbert space $\mathrm{Dom}\mathfrak{s}$. Hence there exists $B \in B(\mathrm{Dom}\mathfrak{s})$ such that

$$(\mathfrak{t}+1)(w,v) = (w|Bv)_{\mathfrak{s}} = (w|Bv) + \mathfrak{s}(w,Bv).$$

We have

$$\|v\|_{\mathfrak{s}}^2 = \mathrm{Re}(\mathfrak{t}+1)(v) = \mathrm{Re}(v|Bv)_{\mathfrak{s}} \leq \|Bv\|_{\mathfrak{s}}\|v\|_{\mathfrak{s}}.$$

Hence $\|v\|_{\mathfrak{s}} \leq \|Bv\|_{\mathfrak{s}}$. Therefore, $\mathrm{Ran}\, B$ is closed.

If w is orthogonal in $\mathrm{Dom}\mathfrak{s}$ to $\mathrm{Ran}\, B$, then

$$\|w\|_{\mathfrak{s}}^2 = \mathrm{Re}(w|Bw)_{\mathfrak{s}} = 0.$$

Hence, $w = 0$. Therefore, B is invertible. □

Proof of Theorem 4.13. Let S denote the operator associated with $\mathfrak{s}$. Then $T := S(B+1) - 1$ with the domain $\mathrm{Dom}T = \mathrm{Dom}S$ satisfies the conditions of the theorem. □

4.8 Perturbations of sectorial forms

Theorem 4.15 *Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be sectorial forms.*

(1) $\mathfrak{t}_1 + \mathfrak{t}_2$ *is also a sectorial form.*

(2) *If $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are closed, then $\mathfrak{t}_1 + \mathfrak{t}_2$ is closed as well.*

(3) *If $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are closable, then $\mathfrak{t}_1 + \mathfrak{t}_2$ is closable as well and* $(\mathfrak{t}_1 + \mathfrak{t}_2)^{\mathrm{cl}} \subset \mathfrak{t}_1^{\mathrm{cl}} + \mathfrak{t}_2^{\mathrm{cl}}$.

Definition 4.16 *Let* $\mathfrak{p}$, $\mathfrak{s}$ *be forms and let* $\mathfrak{s}$ *be positive. We say that* $\mathfrak{p}$ *is* $\mathfrak{s}$*-bounded iff* $\mathrm{Dom}\mathfrak{s} \subset \mathrm{Dom}\mathfrak{p}$ *and*

$$b := \inf_{c>0} \sup_{v\in \mathrm{Dom}\, s} \frac{|\mathfrak{p}(v)|}{\mathfrak{s}(v) + c\|v\|^2} < \infty.$$

The number b *is called the* $\mathfrak{s}$*-bound of* $\mathfrak{p}$.

Theorem 4.17 *Let* $\mathfrak{t}$ *be sectorial and let* $\mathfrak{p}$ *be* $\mathrm{Re}\mathfrak{t}$*-bounded with the* $\mathrm{Re}\mathfrak{t}$*-bound* < 1. *Then*

(1) *The form* $\mathfrak{t}+\mathfrak{p}$ *(with the domain* $\mathrm{Dom}\mathfrak{t}$*) is sectorial as well.*

(2) $\mathfrak{t}$ *is closed* $\Leftrightarrow$ $\mathfrak{t}+\mathfrak{p}$ *is closed.*

(3) $\mathfrak{t}$ *is closable* $\Leftrightarrow$ $\mathfrak{t}+\mathfrak{p}$ *is closable, and then* $\mathrm{Dom}(\mathfrak{t}+\mathfrak{p})^{\mathrm{cl}} = \mathrm{Dom}\mathfrak{t}^{\mathrm{cl}}$.

Proof. Let us prove (1). For some $b < 1$, we have

$$|\mathfrak{p}(v)| \leq b\mathrm{Re}\mathfrak{t}(v) + c\|v\|^2.$$

Hence

$$\begin{aligned}\mathrm{Im}(\mathfrak{t}+\mathfrak{p})(v) &\leq |\mathrm{Im}\mathfrak{t}(v)| + |\mathrm{Im}\mathfrak{p}(v)| \leq (\tan\theta + b)\mathrm{Re}\mathfrak{t}(v) + c\|v\|^2,\\ \mathrm{Re}(\mathfrak{t}+\mathfrak{p})(v) &\geq \mathrm{Re}\mathfrak{t}(v) - |\mathrm{Im}\mathfrak{p}(v)| \geq (1-b)\mathrm{Re}\mathfrak{t}(v) - c\|v\|^2. \end{aligned}\tag{4.17}$$

Hence,

$$|\mathrm{Im}(\mathfrak{t}+\mathfrak{p})(v)| \leq (1-b)^{-1}(\tan\theta + b)\left(\mathrm{Re}(\mathfrak{t}+\mathfrak{p})(v) + c\|v\|^2\right) + c\|v\|^2.$$

This proves that $\mathfrak{t}+\mathfrak{p}$ is sectorial.

To see (2) and (3), note that (4.17) and

$$(1+b)\mathrm{Re}\mathfrak{t}(v) + c\|v\|^2 \geq \mathrm{Re}(\mathfrak{t}+\mathfrak{p})(v)$$

prove that the norms $\|\cdot\|_{\mathfrak{t}}$ and $\|\cdot\|_{\mathfrak{t}+\mathfrak{p}}$ are equivalent. □

4.9 Friedrichs extensions

Theorem 4.18 *Let* T *be a sectorial operator. Then the form* $\mathfrak{t}(w,v) := (w|Tv)$ *is closable.*

Proof. It suffices to assume that the tip of the sector of $\mathfrak{t}$ is 0. Suppose that $w_n \in \mathrm{Dom}T = \mathrm{Dom}\mathfrak{t}$, $w_n \to 0$, $\lim_{n,m\to\infty} \mathfrak{t}(w_n - w_m) = 0$. Then

$$\begin{aligned}|\mathfrak{t}(w_n)| &\leq |\mathfrak{t}(w_n - w_m, w_n)| + |\mathfrak{t}(w_m, w_n)|\\ &\leq (1+\tan\theta)(\mathrm{Re}\mathfrak{t}(w_n))^{\frac12}(\mathrm{Re}\mathfrak{t}(w_n - w_m))^{\frac12} + (w_m|Tw_n).\end{aligned}$$

For any $\epsilon > 0$ there exists N such that for $n, m > N$ we have $\mathrm{Re}\mathfrak{t}(w_n - w_m) \leq \epsilon^2$. Besides, $\lim_{m\to\infty}(w_m|Tw_n) = 0$. Therefore,

$$|\mathfrak{t}(w_n)| \leq \epsilon(1+\tan\theta)|\mathfrak{t}(w_n)|^{1/2}.$$

Hence $\mathfrak{t}(w_n) \to 0$. □

Thus there exists a unique m-sectorial operator T_{Fr} associated with the form $\mathfrak{t}^{\mathrm{cl}}$. The operator T_{Fr} is called the Friedrichs extension of T.

5 Aronszajn-Donoghue and Friedrichs Hamiltonian and their renormalization

5.1 Aronszajn Donoghue Hamiltonians

Let H_0 be a self-adjoint operator on $\mathcal{H}$, $h \in \mathcal{H}$ and $\lambda \in \mathbb{R}$.

$$H := H_0 + \lambda |h)(h|, \tag{5.18}$$

is a rank one perturbation of H_0. We will call (5.18) the Aronszajn Donoghue Hamiltonian.

We would like to describe how to define the Aronszajn-Donoghue Hamiltonian if h is not necessarily a bounded functional on $\mathcal{H}$. It will turn out that it is natural to consider 3 types of h:

$$\text{I. } h \in \mathcal{H}, \qquad \text{II. } h \in \mathcal{H}_{-1} \backslash \mathcal{H}, \qquad \text{III. } h \in \mathcal{H}_{-2} \backslash \mathcal{H}_{-1}, \tag{5.19}$$

where by $\mathcal{H}_{-n}$ we denoted the usual scale of spaces associated to the operator H_0, that is $\mathcal{H}_{-n} := \langle H_0 \rangle^{n/2} \mathcal{H}$, where $\langle H_0 \rangle := (1 + H_0^2)^{1/2}$.

Clearly, in the case I H is self-adjoint on $\mathrm{Dom} H_0$. We will see that in the case II one can easily define H as a self-adjoint operator, but its domain is no longer equal to $\mathrm{Dom} H_0$. In the case III, strictly speaking, the formula (5.18) does not make sense. Nevertheless, it is possible to define a *renormalized Aronszajn-Donoghue Hamiltonian.* To do this one needs to *renormalize the parameter* λ. This procedure resembles the *renormalization of the charge* in quantum field theory.

Consider first the case I. We can compute its resolvent. In fact, for $z \notin \mathrm{sp} H_0$ we define an analytic function

$$g(z) := -\lambda^{-1} + (h|(z - H_0)^{-1} h). \tag{5.20}$$

Then for $z \in \Theta := \{z \in \mathbb{C} \backslash \mathrm{sp} H_0 \ : \ g(z) \neq 0\}$ and $\lambda \neq 0$, the resolvent of the operator H is given by Krein's formula

$$R(z) = (z - H_0)^{-1} - g(z)^{-1} (z - H_0)^{-1} |h)(h| (z - H_0)^{-1}. \tag{5.21}$$

For $\lambda = 0$, we set $\Theta = \mathbb{C} \backslash \mathrm{sp} H_0$ and clearly

$$R(z) = (z - H_0)^{-1}. \tag{5.22}$$

The following theorem will describe how to define the Aronszajn-Donoghue Hamiltonian in all the cases I, II and III:

Theorem 5.1 *Assume that:*
(A) $h \in \mathcal{H}_{-1}$, $\lambda \in \mathbb{R} \cup \{\infty\}$ and let $R(z)$ be given by (5.22) or (5.21) with $g(z)$ given by (5.20),
or
(B) $h \in \mathcal{H}_{-2}$, $\gamma \in \mathbb{R}$ and let $R(z)$ be given by (5.21) with $g(z)$ given by

$$g(z) := \gamma + \left(h | ((z - H_0)^{-1} + H_0 (1 + H_0^2)^{-1}) h \right).$$

Then, for all $z \in \Theta$,

1. *$R(z)$ is a bounded operator which fulfills the first resolvent formula;*
2. *$\mathrm{Ker} R(z) = \{0\}$, unless $h \in \mathcal{H}$ and $\lambda = \infty$;*
3. *$\mathrm{Ran}\, R(z)$ is dense in $\mathcal{H}$, unless $h \in \mathcal{H}$ and $\lambda = \infty$;*
4. *$R(z)^* = R(\overline{z})$.*

Hence, except for the case $h \in \mathcal{H}$, $\lambda = \infty$, there exists a unique densely defined self-adjoint operator H such that $R(z)$ is the resolvent of H.

Another way to define H for the case $h \in \mathcal{H}_{-2}$ is the cut-off method. For all $k \in \mathbb{N}$ we define h_k as in (5.35) and fix the *running coupling constant* by

$$-\lambda_k^{-1} := \gamma + (h_k|H_0(1+H_0^2)^{-1}h_k)$$

and set the cut-off Hamiltonian to be

$$H_k := H_0 + \lambda_k |h_k)(h_k|. \tag{5.23}$$

Then the resolvent for H_k is given by

$$R_k(z) = (z - H_0)^{-1} + g_k(z)^{-1}(z - H_0)^{-1}|h_k)(h_k|(z - H_0)^{-1}, \tag{5.24}$$

where

$$g_k(z) := -\lambda_k^{-1} + \left(h_k|(z - H_0)^{-1}h_k\right). \tag{5.25}$$

Note that λ_k is chosen in such a way that the renormalization condition $\frac{1}{2}\left(g_k(\mathrm{i}) + g_k(-\mathrm{i})\right) = \gamma$. holds. It is easy to see that if H_0 is bounded from below, then $\lim_{k\to\infty} \lambda_k = 0$. Again, the cut-off Hamiltonian converges to the renormalized Hamiltonian:

Theorem 5.2 *Assume that* $h \in \mathcal{H}_{-2}$. *Then* $\lim\limits_{k\to\infty} R_k(z) = R(z)$.

Let us assume that h is cyclic. Then the support of the spectral measure of h wrt H_0 is $\mathrm{sp}H_0$. If $g(\beta) = 0$ and $\beta \notin \mathrm{sp}H_0$, then H has an eigenvalue at β and the corresponding eigenprojection equals

$$1_{\{\beta\}}(H) = (h|(\beta - H_0)^{-2}h)^{-1}(\beta - H_0)^{-1}|h)(h|(\beta - H_0)^{-1}.$$

5.2 Aronszajn-Donoghue Hamiltonians and extensions of Hermitian operators

Let H_0 be as above and $h \in \mathcal{H}_{-2}\backslash\mathcal{H}$. Define $H_{\min}$ to be the restriction of H_0 to

$$\mathrm{Dom}(H_{\min}) := \{v \in \mathrm{Dom}(H_0) = \mathcal{H}_2 \ : \ (h|v) = 0\}.$$

Then $H_{\min}$ is a closed Hermitian operator. Set $H_{\max} := H_{\min}^*$. Then

$$\mathrm{Dom}(H_{\max}) = \mathrm{Span}\{\mathrm{Dom}H_{\min},\ (\mathrm{i} - H_0)^{-1}h,\ (-\mathrm{i} - H_0)^{-1}h, \}.$$

Note that $\mathrm{Ker}(H_{\max} \pm \mathrm{i})$ is spanned by

$$v_\pm := (\pm\mathrm{i} - H_0)^{-1}h.$$

Thus the indices of defect of $H_{\min}$ are $(1, 1)$.

The operators H_γ discussed in the previous subsection are self-adjoint extensions of $H_{\min}$. To obtain H_γ it suffices to increase the domain of $H_{\min}$ by adding the vector

$$\theta_\gamma(\mathrm{i} - H_0)^{-1}h - \overline{\theta}_\gamma(\mathrm{i} + H_0)^{-1}h,$$

where $\theta_\gamma := \frac{\gamma + (h|H_0(1+H_0^2)-1h)}{\gamma - \mathrm{i}(h|(1+H_0^2)^{-1}h)}$.

5.3 Aronszajn-Donoghue Hamiltonians and extensions of positive forms

Assume now in addition that H_0 is positive.

Consider the positive form $\mathfrak{h}_{\min}$ associated with $H_{\min}$. Thus $\mathfrak{h}(v, v) = (v|H_{\min}v) = (v|H_0 v)$ with the domain $\mathrm{Dom}(\mathfrak{h}_{\min}) := \mathrm{Dom}H_{\min}$.

Assume first that $h \in \mathcal{H}_{-1}$. The Friedrichs extension of $H_{\min}$ is H_λ with $\lambda = \infty$. The closure of the form $\mathfrak{h}_{\min}$ has the domain $\{v \in \mathcal{H}_1 \ : \ (h|v) = 0\}$.

Assume now that $h \in \mathcal{H}_{-2}\backslash\mathcal{H}_{-1}$. Then the Friedrichs extension of $H_{\min}$ equals H_0.

5.4 Friedrichs Hamiltonian

Let H_0 be again a self-adjoint operator on the Hilbert space $\mathcal{H}$. Let $\epsilon \in \mathbb{R}$ and $h \in \mathcal{H}$. The following operator on the Hilbert space $\mathbb{C} \oplus \mathcal{H}$ is often called the Friedrichs Hamiltonian:

$$G := \begin{bmatrix} \epsilon & (h| \\ |h) & H_0 \end{bmatrix}. \tag{5.26}$$

Recall that expression the operators $(h|$ and $|h)$ are defined by

$$\begin{aligned} \mathcal{H} \ni v &\mapsto (h|v := (h|v) \in \mathbb{C}, \\ \mathbb{C} \ni \alpha &\mapsto |h)\alpha := \alpha h \in \mathcal{H}. \end{aligned} \tag{5.27}$$

We would like to describe how to define the Friedrichs Hamiltonian if h is not necessarily a bounded functional on $\mathcal{H}$. It will turn out that it is natural to consider 3 types of h:

$$\text{I. } h \in \mathcal{H}, \qquad \text{II. } h \in \mathcal{H}_{-1} \backslash \mathcal{H}, \qquad \text{III. } h \in \mathcal{H}_{-2} \backslash \mathcal{H}_{-1}, \tag{5.28}$$

Clearly, in the case I G is self-adjoint on $\mathbb{C} \oplus \mathrm{Dom} H_0$. We will see that in the case II one can easily define G as a self-adjoint operator, but its domain is no longer equal to $\mathbb{C} \oplus \mathrm{Dom} H_0$. In the case III, strictly speaking, the formula (5.26) does not make sense. Nevertheless, it is possible to define a *renormalized Friedrichs Hamiltonian.* To do this one needs to *renormalize the parameter* ϵ. This procedure resembles the *renormalization of mass* in quantum field theory. Let us first consider the case $h \in \mathcal{H}$. As we said earlier, the operator G with $\mathrm{Dom} G = \mathbb{C} \oplus \mathrm{Dom} H_0$ is self-adjoint. It is well known that the resolvent of G can be computed exactly. In fact, for $z \notin \mathrm{sp} H_0$ define the analytic function

$$g(z) := \epsilon + (h|(z - H_0)^{-1} h). \tag{5.29}$$

Then for $z \in \Omega := \{z \in \mathbb{C} \backslash \mathrm{sp} H_0 \ : \ g(z) - z \neq 0\}$ the resolvent $Q(z) := (z - G)^{-1}$ is given by

$$\begin{aligned} Q(z) \ &= \ \begin{bmatrix} 0 & 0 \\ 0 & (z - H_0)^{-1} \end{bmatrix} \\ &\quad + \Big(z - g(z) \Big)^{-1} \begin{bmatrix} 1 & (h|(z - H_0)^{-1} \\ (z - H_0)^{-1}|h) & (z - H_0)^{-1}|h)(h|(z - H_0)^{-1} \end{bmatrix}. \end{aligned} \tag{5.30}$$

Theorem 5.3 *Assume that:*
(A) $h \in \mathcal{H}_{-1}$, $\epsilon \in \mathbb{R}$ *and let* $Q(z)$ *be given by (5.30) with* $g(z)$ *defined by (5.29),*
or
(B) $h \in \mathcal{H}_{-2}$, $\gamma \in \mathbb{R}$ *and let* $Q(z)$ *be given by (5.30) with* $g(z)$ *defined by*

$$\begin{aligned} g(z) := \ &\gamma + \big(h|((z - H_0)^{-1} + H_0(1 + H_0^2)^{-1}) h \big) \\ = \ &\gamma + \Big(h|\big(\tfrac{i - z}{2(z - H_0)(i - H_0)} - \tfrac{i + z}{2(z - H_0)(-i - H_0)} \big) h \Big) \end{aligned} \tag{5.31}$$

Then for all $z \in \Omega$ *:*

1. $Q(z)$ *is a bounded operator which fulfills the first resolvent formula (in the terminology of [Ka],* $Q(z)$ *is a pseudoresolvent);*
2. $\mathrm{Ker} Q(z) = \{0\}$*;*
3. $\mathrm{Ran}\, Q(z)$ *is dense in* $\mathbb{C} \oplus \mathcal{H}$*;*

(4) $Q(z)^* = Q(\overline{z})$.

Therefore, by [Ka], there exists a unique densely defined self-adjoint operator G such that $Q(z) = (z - G)^{-1}$. More precisely, for any $z_0 \in \Omega$, $\mathrm{Dom}G = \mathrm{Ran}\, Q(z_0)$*, and if* $\varphi \in \mathrm{Ran}\, Q(z_0)$ *and* $Q(z_0)\psi = \varphi$*, then*

$$G\varphi := -\psi + z_0 Q(z_0)\psi,$$

Proof. Let $z \in \Omega$. It is obvious that $Q(z)$ is bounded and satisfies (4). We easily see that both in the case (A) and (B) the function $g(z)$ satisfies

$$g(z_1) - g(z_2) = -(z_1 - z_2)(h|(z_1 - H_0)^{-1}(z_2 - H_0)^{-1}|h). \tag{5.32}$$

Direct computations using (5.32) show the first resolvent formula.

Let $(\alpha, f) \in \mathbb{C} \oplus \mathcal{H}$ be such that $(\alpha, f) \in \mathrm{Ker}Q(z)$. Then

$$0 = (z - g(z))^{-1}\Big(\alpha + (h|(z - H_0)^{-1}f)\Big), \tag{5.33}$$

$$0 = (z - H_0)^{-1}f + (z - H_0)^{-1}h(z - g(z))^{-1}\Big(\alpha + (h|(z - H_0)^{-1}f)\Big). \tag{5.34}$$

Inserting (5.33) into (5.34) we get $0 = (z - H_0)^{-1}f$ and hence $f = 0$. Now (5.33) implies $\alpha = 0$, so $\mathrm{Ker}Q(z) = \{0\}$.

Using (2) and (4) we get $(\mathrm{Ran}\, Q(z))^{\perp} = \mathrm{Ker}Q(z)^* = \mathrm{Ker}Q(\overline{z}) = \{0\}$. Hence 3) holds. □

Let $h \in \mathcal{H}_{-2}$ and $\gamma \in \mathbb{R}$. Let us impose a cut-off on h. For $k \in \mathbb{N}$ we define

$$h_k := 1_{[-k,k]}(H_0)\, h, \tag{5.35}$$

where $1_{[-k,k]}(H_0)$ is the spectral projection for H_0 associated with the interval $[-k, k] \subset \mathbb{R}$. Note that $h_k \in \mathcal{H}$ and hence both $(h_k|$ and $|h_k)$ are well defined bounded operators. Set

$$\epsilon_k := \gamma + (h_k|H_0(1 + H_0^2)^{-1}h_k).$$

For all $k \in \mathbb{N}$, the cut-off Friedrichs Hamiltonian

$$G_k := \begin{bmatrix} \epsilon_k & (h_k| \\ |h_k) & H_0 \end{bmatrix}$$

is well defined and we can compute its resolvent, $Q_k(z) := (z - G_k)^{-1}$:

$$\begin{aligned} Q_k(z) \quad = \quad & \begin{bmatrix} 0 & 0 \\ 0 & (z - H_0)^{-1} \end{bmatrix} \\ & + \Big(z - g_k(z)\Big)^{-1} \begin{bmatrix} 1 & (h_k|(z - H_0)^{-1} \\ (z - H_0)^{-1}|h_k) & (z - H_0)^{-1}|h_k)(h_k|(z - H_0)^{-1} \end{bmatrix}. \end{aligned} \tag{5.36}$$

where

$$g_k(z) := \epsilon_k + (h_k|(z - H_0)^{-1}h_k). \tag{5.37}$$

Note that ϵ_k is chosen such a way that the following *renormalization condition* is satisfied: $\frac{1}{2}\left(g_k(\mathrm{i}) + g_k(-\mathrm{i})\right) = \gamma$. Let us also mention that if H_0 is bounded from below, then $\lim_{k\to\infty} \epsilon_k = \infty$.

Theorem 5.4 *Assume that* $h \in \mathcal{H}_{-2}$*. Then* $\lim\limits_{k\to\infty} Q_k(z) = Q(z)$*, where* $Q(z)$ *is given by (5.30) and* $g(z)$ *is given by (5.31).*

Proof. The proof is obvious if we note that $\lim_{k\to\infty}\|(z-H_0)^{-1}h-(z-H_0)^{-1}h_k\|=0$ and $\lim_{k\to\infty}g_k(z)=g(z)$.
□

Thus the cut-off Friedrichs Hamiltonian is norm resolvent convergent to the renormalized Friedrichs Hamiltonian.

Let us assume that h is cyclic. Then the support of the spectral measure of h wrt H_0 is spH_0. If $\beta=g(\beta)=0$ and $\beta\notin\mathrm{sp}H_0$, then G has an eigenvalue at β. The corresponding projection equals

$$1_\beta(G)=(1+(h|(\beta-H_0)^{-2}|h))^{-1}\left[\begin{array}{cc}1 & (h|(\beta-H_0)^{-1}\\ (\beta-A)^{-1}|h) & (\beta-H_0)^{-1}|h)(h|(\beta-H_0)^{-1}\end{array}\right].$$

6 Discrete and essential spectrum

6.1 Extended discrete and essential spectrum

Let $\mathcal{X}$ be a Banach space and A a closed operator on $\mathcal{X}$. Recall that we defined the discrete and essential spectra of A denoted by $\mathrm{sp}_{\mathrm{d}}A$ and $\mathrm{sp}_{\mathrm{ess}}A$.

We say that ∞ belongs to the extended discrete spectrum iff there is a decomposition of $\mathcal{X}=\mathcal{X}_0\oplus\mathcal{X}_1$ into the direct sum of two closed subspaces such that $\mathcal{X}_1$ is nonzero finite dimensional, Dom$A=\mathcal{X}_0$ and A maps $\mathcal{X}_0$ into itself and A restricted to $\mathcal{X}_0$ is bounded. Equivalently, ∞ is a discrete point in $\mathrm{sp}^{\mathrm{ext}}A$ and $1_{\{\infty\}}(A)$ is finite dimensional. The discrete extended spectrum is denoted by $\mathrm{sp}_{\mathrm{d}}^{\mathrm{ext}}(A)$. The essential extended spectrum is defined as

$$\mathrm{sp}_{\mathrm{ess}}^{\mathrm{ext}}A:=\mathrm{sp}^{\mathrm{ext}}A\backslash\mathrm{sp}_{\mathrm{d}}^{\mathrm{ext}}A.$$

Theorem 6.1 *Let $z_0\in\mathrm{rs}A$. Then*

$$\mathrm{sp}_{\mathrm{ess}}^{\mathrm{ext}}(z_0-A)^{-1}=(z_0-\mathrm{sp}_{\mathrm{ess}}^{\mathrm{ext}}A)^{-1},\quad \mathrm{sp}_{\mathrm{d}}^{\mathrm{ext}}(z_0-A)^{-1}=(z_0-\mathrm{sp}_{\mathrm{d}}^{\mathrm{ext}}A)^{-1}.$$

6.2 Operators with a compact resolvent

Theorem 6.2 *Let A be an operator with a non-empty resolvent set. Then the following conditions are equivalent:*

(1) *$(z_0-A)^{-1}$ is compact for some $z_0\in\mathrm{rs}A$.*

(2) *$(z-A)^{-1}$ is compact for all $z\in\mathrm{rs}A$.*

Proof. We use the resolvent equation

$$(z-A)^{-1}=(z_0-A)^{-1}\left(1-(z-z_0)(z-A)^{-1}\right).$$

□

When the conditions of Theorem 6.2 are satisfied, then we say that the operator A has a compact resolvent.

Theorem 6.3 (1) *Let A be normal. Then A has a compact resolvent iff $\mathrm{sp}A=\mathrm{sp}_{\mathrm{d}}A$.*

(2) *Let A be bounded from below and self-adjoint. Then A has a compact resolvent iff $\mu_n(A)\to\infty$.*

Theorem 6.4 *Let $f,g\in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$, $\lim_{|x|\to\infty}f(x)=\infty$ and $\lim_{|x|\to\infty}g(x)=\infty$. Then*

$$H:=f(x)+g(D)$$

has a compact resolvent.

Proof. Clearly, the functions f, g are bounded from below. Fix $\epsilon > 0$. For $r > 0$,let

$$H_r := f(x) + \min(g(D), r).$$

Then for $z \in \mathbb{C}\backslash\mathbb{R}$ the operator $(\min(g(D), r) - r)(z - f(x) - r)^{-1}$ is compact. Hence also the operator

$$(z - H_r)^{-1} - (z - f(x) - r)^{-1} = (z - H_r)^{-1}(\min(g(D), r) - r)(z - f(x) - r)^{-1}$$

is compact. Thus

$$\mathrm{sp}_{\mathrm{ess}}(H_r) = \mathrm{sp}_{\mathrm{ess}}(f(x) + r) \subset [r + \inf f, \infty[.$$

Therefore, there exists N such that for $n > N$

$$\mu_n(H_r) \geq r - \epsilon + \inf f.$$

But $H_r \leq H$. Hence

$$\mu_n(H_r) \leq \mu_n(H).$$

Therefore, for $n > N$, we have $\mu_n(H) \geq R - \epsilon + \inf f$. Thus $\mu_n(H) \to \infty$. □

6.3 Stability of essential spectrum

Theorem 6.5 *Let $\mathcal{V}$ be a Hilbert space and $A \in B(\mathcal{V})$ be normal. Then $\lambda \in \mathrm{sp}_{\mathrm{ess}} A$ iff there exits a sequence of vectors v_n such that*

$$\|v_n\| = 1, \quad \mathrm{w-}\lim_{n\to\infty} v_n = 0, \quad \lim_{n\to\infty} (A - \lambda)v_n = 0. \tag{6.38}$$

Proof. ⇒ We know that for any n $\dim 1_{B(\lambda,\frac{1}{n})}(A) = \infty$. Therefore, we can find an orthonormal system $v_1, v_2, \ldots$ such that $v_n \in \mathrm{Ran}\, 1_{B(\lambda,\frac{1}{n})}(A)$. The sequence $v_1, v_2, \ldots$ satisfies (6.38).

⇐ Let $\epsilon > 0$. We have

$$(1 - 1_{B(\lambda,\epsilon)}(A))v_n = (1 - 1_{B(\lambda,\epsilon)}(A))(A - \lambda)^{-1}(A - \lambda)v_n \to 0.$$

Hence

$$c_n := \|1_{B(\lambda,\epsilon)}(A)v_n\| \to 1.$$

Let

$$\tilde{v}_n := \frac{1}{c_n} 1_{B(\lambda,\epsilon)}(A)v_n.$$

Then $\|\tilde{v}_n\| = 1$, $\mathrm{w-}\lim_{n\to\infty} \tilde{v}_n = 0$ and $\tilde{v}_n \in \mathrm{Ran}\, 1_{B(\lambda,\epsilon)}(A)$. Hence $\mathrm{Span}\{\tilde{v}_1, \tilde{v}_2, \ldots\}$ is infinite dimensional. Thus, $1_{B(\lambda,\epsilon)}(A)$ is infinite dimensional. □

Definition 6.6 *A sequence of vectors v_n satisfying the conditions of the above theorem will be called a Weyl sequence for λ and the operator A.*

Theorem 6.7 (Weyl) *Let $A, B \in B(\mathcal{V})$ be normal and let $B - A$ be compact. Then $\mathrm{sp}_{\mathrm{ess}} A = \mathrm{sp}_{\mathrm{ess}} B$.*

Proof. Assume that $\lambda \in \mathrm{sp}_{\mathrm{ess}} A$. Then there exists a Weyl sequence $v_1, v_2, \ldots$ for λ and the operator A. We have $\lim_{n\to\infty}(B - A)v_n = 0$. Hence $v_1, v_2, \ldots$ is a Weyl sequence for λ and the operator B. □

Theorem 6.8 *Assume that* A, B *are normal operators such that for some* $z_0 \notin \mathrm{sp}A \cup \mathrm{sp}B$ *the operator* $(z_0 - A)^{-1} - (z_0 - B)^{-1}$ *is compact. Then*

$$\mathrm{sp}_{\mathrm{ess}}A = \mathrm{sp}_{\mathrm{ess}}B.$$

Proof. By the Weyl theorem $\mathrm{sp}_{\mathrm{ess}}(z_0 - A)^{-1} = \mathrm{sp}_{\mathrm{ess}}(z_0 - B)^{-1}$. Then we use Theorem 6.1 to normal operators A and B. □

Theorem 6.9 *Let* $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$, $\lim_{|x|\to\infty} f(x) = \infty$ *and* $g \in L^\infty(\mathbb{R}^d)$, $\lim_{|x|\to\infty} g(x) = 0$. *Then*

$$\mathrm{sp}_{\mathrm{ess}}(f(D) + g(x)) = \mathrm{sp}_{\mathrm{ess}}(f(D)).$$

Proof. The operator $g(x)(z_0 - f(D))^{-1}$ is compact. Hence, the operator

$$(z_0 - f(D) - g(x))^{-1} - (z_0 - f(D))^{-1} = (z_0 - f(D) - g(x))^{-1} g(x)(z_0 - f(D))^{-1}$$

is compact as well. We can thus use Theorem 6.8. □

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