ASYMPTOTIC COMPLETENESS
OF N-BODY SCATTERING

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In my opinion, scattering theory for $N$-body Schrödinger operators is one of the greatest successes of 20th century mathematical physics.

On the physical side, we have a rigorous framework that explains why nonrelativistic matter is built out of well defined clusters of nuclei and electrons, such as atoms, ions, molecules.

On the mathematical side, we have a deep analysis of a large family of nontrivial operators with continuous spectrum, combining ideas from classical and quantum mechanics.
A single quantum particle in an external potential is described by the Hilbert space $L^2(\mathbb{R}^d)$ and the Schrödinger Hamiltonian

$$H = H_0 + V(x),$$

where

$$H_0 = \frac{p^2}{2m}, \quad p = \frac{1}{i} \partial_x.$$

A typical example of a potential is

$$V(x) = \frac{c}{|x|}.$$
THEOREM. Assume that $V(x)$ is short range, that is,

$$|V(x)| \leq c\langle x \rangle^{-\mu_s}, \quad \mu_s > 1.$$ 

Then there exist wave (Møller) operators

$$\Omega^\pm := \text{s-}\lim_{t \to \pm \infty} e^{itH} e^{-itH_0},$$

they are isometric, they intertwine the free and full Hamiltonian:

$$\Omega^\pm H_0 = H \Omega^\pm,$$

and they are complete:

$$\Omega^\pm \Omega^{\pm*} = 1_l_c(H).$$
THEOREM. Assume that $V(x)$ is long range, that is,

$$V(x) = V_1(x) + V_s(x),$$

where $V_s(x)$ is short range and

$$|\partial_x^\alpha V_1(x)| \leq c_\alpha \langle x \rangle^{-|\alpha|-\mu_1}, \quad \mu_1 > 0, \quad \alpha \in \mathbb{N}^d.$$

Then there exists a function $(t, \xi) \mapsto S_t(\xi)$ and modified Møller operators

$$\Omega^\pm := \text{s- \lim}_{t \to \pm\infty} e^{itH} e^{-iS_t(p)},$$

which satisfy the same properties as those stated for the short-range case.
Thus the Hilbert space is the direct sum of bound states and of scattering states – states which evolve for large times as free waves. One can define the scattering operator,

$$S := \Omega^+ \Omega^{-*},$$

which is unitary. The integral kernel of $S$ defines scattering amplitudes. The square of the absolute value of a scattering amplitude is the scattering cross-section describing the probability of a scattering process.

The most difficult part of the above theorems is to prove that the range of (modified) wave operators fills the whole continuous spectral space of $H$. This is called asymptotic completeness (AC).
2 interacting quantum particles are described by the Hilbert space $L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d) \simeq L^2(\mathbb{R}^{2d})$ and the Hamiltonian

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(x_1 - x_2).$$

Introduce the center-of-mass coordinate $x_{12} := \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$ and the relative coordinate $x^{12} := x_2 - x_1$. The Hilbert space factorizes

$$L^2(\mathbb{R}^{2d}) = L^2(X_{12}) \otimes L^2(X^{12}).$$
Let $m_{12} := m_1 + m_2$ be the total mass and $m^{12} := (m_1^{-1} + m_2^{-1})^{-1}$ be the reduced mass. Then we can write

$$H = \frac{p_{12}^2}{2m_{12}} + H^{12},$$

where

$$H^{12} := \frac{(p^{12})^2}{2m^{12}} + V(x^{12}).$$

Thus the problem of two interacting particles is reduced to a single particle in an external potential.
$N$ interacting quantum particles are described by the Hilbert space

$$ \otimes_{i=1}^{N} L^2(\mathbb{R}^d) \simeq L^2(X), $$

where $X := \mathbb{R}^{Nd}$, and the Hamiltonian is

$$ H := \sum_{j=1}^{N} \frac{p_j^2}{2m_j} + \sum_{1 \leq i < j \leq N} V_{ij}(x_i - x_j). $$

A typical potential is

$$ V_{ij}(x_i - x_j) = \frac{Z_i Z_j e^2}{4\pi|x_i - x_j|}. $$
A cluster decomposition is a partition of \(\{1, \ldots, N\}\) into clusters:

\[
a = \{c_1, \ldots, c_k\}.
\]

The Hamiltonian of a cluster \(c\) is

\[
H_c := \sum_{j \in c} \frac{p_j^2}{2m_j} + \sum_{i,j \in c} V_{ij}(x_i - x_j).
\]

The Hamiltonian of a cluster decomposition \(a\) is

\[
H_a = H_{c_1} + \cdots + H_{c_k}.
\]
Note that cluster decompositions have a natural order. In particular, there is a minimal cluster decomposition, where all clusters are 1-element. Every pair determines a cluster decomposition.

Define the collision plane of $a$ as

$$X_a := \{(x_1, \ldots, x_N) \in \mathbb{R}^{Nd} : (ij) \leq a \Rightarrow x_i = x_j\}.$$ 

Consider the quadratic form on $X$

$$\sum \frac{m_i}{2} x_i^2.$$ 

Let $X^a$ denote the internal plane of $a$, defined as the orthogonal complement of $X_a$ wrt this form. We will write $x \mapsto x_a$ and $x \mapsto x^a$ for the orthogonal projections onto $X_a$ and $X^a$. 
We have
\[ X = X_a \oplus X^a, \quad X^a = X^{c_1} \oplus \cdots \oplus X^{c_k}. \]

Therefore,
\[ L^2(X) = L^2(X_a) \otimes L^2(X^a), \quad L^2(X^a) = L^2(X^{c_1}) \otimes \cdots \otimes L^2(X^{c_1}). \]

The cluster Hamiltonian decomposes:
\[ H_a = T_a + H^a, \quad H^a = H^{c_1} + \cdots + H^{c_k}. \]
Introduce
\[ \mathcal{H}^a := \text{Ran} \mathbb{1}_p(H^a) \simeq \text{Ran} \mathbb{1}_p(H^{c_1}) \otimes \cdots \otimes \text{Ran} \mathbb{1}_p(H^{c_k}). \]

Let
\[ E^a := H^a \bigg|_{\mathcal{H}^a} = H^{c_1} \bigg|_{\mathcal{H}^{c_1}} + \cdots + H^{c_k} \bigg|_{\mathcal{H}^{c_k}} \]
be the operator describing the bound state energies of clusters. Let
\[ J^a : L^2(X_a) \otimes \mathcal{H}^a \to L^2(X) \]
be the embedding of bound states of clusters into the full Hilbert space.
THEOREM. Assume that the potentials \( V_{ij} \) are short range. Then for any cluster decomposition \( a \) there exists the corresponding partial wave operator

\[
\Omega_a^{\pm} := \lim_{t \to \pm \infty} e^{itH} J_a e^{-it(T_a + E^a)}. 
\]

\( \Omega_a^{\pm} \) are isometric, they intertwine the cluster and the full Hamiltonian:

\[
\Omega_a^{\pm}(T_a + E^a) = H \Omega_a^{\pm}
\]

and are complete:

\[
\bigoplus_a \text{Ran} \Omega_a^{\pm} = L^2(X).
\]
THEOREM. Assume that the potentials $V_{ij}$ are long range with

$$\mu_1 > \sqrt{3} - 1.$$ 

Then for any cluster decomposition $a$ there exists a function $(t, \xi_a) \mapsto S_{a,t}(\xi_a)$, the corresponding partial modified wave operator

$$\Omega^\pm_a := \lim_{t \to \pm \infty} e^{itH} J_a e^{-i(S_{a,t}(p_a) + tE^a)},$$

which satisfy the same properties as those stated in the short range case.
AC means that all states in $L^2(X)$ can be decomposed into states with a clear physical/chemical interpretation such as atoms, ions and molecules.

We can introduce partial scattering operators

$$S_{ab} := \Omega_a^+\Omega_b^-$$

describing various processes, such as elastic and inelastic scattering, ionization, capture of an electron, chemical reactions.

The partial wave operators $\Omega_a^\pm$ can be organized into

$$\bigoplus_a L^2(X_a) \otimes \mathcal{H}^a \ni (\psi_a) \mapsto \sum_a \Omega_a^\pm \psi_a \in L^2(X),$$

which is unitary. The partial scattering operators $S_{ab}$ arranged in the matrix $[S_{ab}]$ also describe a unitary operator.
2-body scattering theory, including AC in both short- and long-range case, was understood already in the 60’s.

Existence of $N$-body wave operators and the orthogonality of their ranges was established about the same time. What was missing for a long time was Asymptotic Completeness – the fact that the ranges of wave operators span the whole Hilbert space.

Below I review the various methods that were used, more or less successfully, to prove this.
The stationary approach to scattering theory is based on resolvent identities. For example, if \( H = H_0 + V \), then the identity

\[
(z - H)^{-1} = (z - H_0)^{-1} \\
+ (z - H_0)^{-1}V^{1/2}\left(1 - |V|^{1/2}(z - H_0)^{-1}V^{1/2}\right)^{-1}|V|^{1/2}(z - H_0)^{-1}
\]

can be used to prove AC in the 2-body case.
L. Faddeev found a resolvent identity that can be used to study 3-body scattering. A number of other resolvent identities were used (eg. G. Hagedorn’s for 4 bodies). The results about AC with $N \geq 3$ proven using the stationary approach involve implicit assumptions on invertibility of certain complicated operators and on properties of bound and almost-bound states. They also require a very fast decay of potentials and $d \geq 3$.

However, in principle, the stationary approach leads to explicit formulas for scattering amplitudes.
V. Enss introduced time-dependent methods into proofs of AC. In his approach an important tool was the RAGE Theorem saying that for $K$ compact and $\psi \in \text{Ran}\mathbb{I}^c(H)$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \| Ke^{itH} \psi \|^2 dt = 0.$$ 

Enss started with proving the 2-body AC (late 70’s), and managed to prove 3-body AC including the long-range case with $\mu_1 > \sqrt{3} - 1$ (late 80’s).
Let us describe an idea that turned out to be important: One needs to look for observables $A$ such that $i[H, A]$ is in some sense positive. Here is an important example of this idea: 

E. Mourre (1981). Suppose that $E$ is not a threshold (it is not an eigenvalue of $H_a$ for any $a$). Then there exists an interval $I$ around $E$ and $c_0 > 0$ such that

$$1_I(H)i[H, A]1_I(H) \geq c_0 1_I(H),$$

where $A = \sum_i \frac{1}{2}(p_i x_i + x_i p_i)$ is the generator of dilations.

The Mourre estimate has important implications both in the stationary and time-dependent approach.
I.M. Sigal devoted a large part of his research career to $N$-body AC. After working with the stationary approach he switched to the time-dependent approach. Together with A. Soffer he obtained the first proof of the $N$-body AC in the short range case (announced 1985, published 1987). They first used heavily propagation estimates. Below we summarize abstractly the time-dependent version of this technique:

If $\Phi(t)$ is a uniformly bounded observable on a Hilbert space $\mathcal{H}$ and

$$\frac{d}{dt}\Phi(t) + i[H, \Phi(t)] \geq \Psi^*(t)\Psi(t),$$

then

$$\int_1^{\infty} \|\Psi(t)v\|^2 dt < \infty, \quad v \in \mathcal{H}.$$
A new and elegant proof of the $N$-body AC in the short range case was given by G.M. Graf (1989). Just as Sigal-Soffer’s, it was also time-dependent, used propagation estimates and Mourre estimate. It introduced a clever observable, the Graf vector field, whose commutator with $H$ is positive.
First proof of AC in the long range case for any $N$ with $\mu_1 > \sqrt{-1}$ (which includes the physical Coulomb potentials) was given by J.D (announced 1991, published 1993). There exists a monograph J.D and C.Gérard in Springer Tracts and Monographs in Physics, 1997\footnote{http://www.fuw.edu.pl/ derezins/bookn.pdf} about this subject.

In what follows I describe the main steps of the proof. My presentation will stress some additional features of $N$-body scattering, which I find interesting.
First assume the long-range condition on the potentials with

$$\mu_1 > 0.$$  

Following the ideas of the proof of Graf for the short-range case one can show the existence of the so-called asymptotic velocity:

**THEOREM** For any function $f \in C^\infty_c(X)$ there exists limits

$$s_- \lim_{t \to \pm \infty} e^{itH} f \left( \frac{x}{t} \right) e^{-itH}. \quad (\ast)$$

There exists a family of commuting self-adjoint operators $P^\pm$ such that $(\ast)$ equals $f(P^\pm)$. 
Of course, we can replace $H$ with $H^a$ obtaining $P^{a+}$, the asymptotic velocity corresponding to $a$. The following fact follows by arguments involving the Mourre estimate, and is also essentially due to Graf:

**THEOREM** For any $a$

$$\mathbb{1}_{\{0\}}(P^{a+}) = \mathbb{1}^p(H^a).$$
For any $a$ introduce

$$Z_a := X_a \backslash \bigcup_{b \leq a} X_b.$$ 

Then the family $Z_a$ is a partition of $X$. In particular,

$$1 = \sum_{a} 1_{Z_a}(P^+).$$

Now in the short-range case AC follows easily by proving that

$$\lim_{t \to \pm \infty} e^{itH_a} e^{-itH} 1_{Z_a}(P^+)$$

exists and coincides with $\Omega^\pm_a$. 
In the long-range case one needs an additional step.

**THEOREM** Let $\phi = 1_{Z^a}(P^{\pm})\phi$ and $\delta = \frac{2}{2+\mu}$. Then there exists $c$ such that

$$\lim_{t \to \pm \infty} 1(t^{-\delta}|x^a| > c) e^{\mp itH} \phi = 0.$$  

To see that this bound is natural note that Newton’s equation in the potential $V(x) = -|x|^{-\mu}$ at zero energy has trajectories of the form

$$x(t) = ct^{\frac{2}{2+\mu}}.$$
To prove the existence of the modified wave operator we need to show that the variation of the potential that comes from outside of the given cluster decomposition within a wave packet is integrable in time. The variation of the potential can be estimated by

\[(\text{spread of wave packet}) \times (\text{derivative of potential}) \sim t^{2+\mu} \times t^{-1-\mu}.\]

The integrability condition gives

\[\frac{2}{2+\mu} - 1 - \mu < -1,\]

which is solved by \(\mu > \sqrt{3} - 1.\)