# Interference of Fock states in a single measurement

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We study analytically the structure of an arbitrary-order correlation function for a pair of Fock states and prove without any approximations that, in a single measurement of particle positions, interference effects must occur as experimentally observed with Bose-Einstein condensates. We also show that the noise level present in the statistics is slightly lower than for a corresponding measurement of phase states.

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## I. INTRODUCTION

Paul Dirac in his famous textbook on quantum mechanics [1] describes photon interference in the following way. "Suppose we have a beam of light consisting of a large number of photons split up into two components of equal intensity (…). If the two components are now made to interfere, we should require a photon in one component to be able to interfere with one in the other. Sometimes these two photons would have to annihilate one another and other times they would have to produce four photons. This would contradict the conservation of energy (…). Each photon then interferes only with itself. Interference between two different photons never occurs." This view has been criticized [2] for applying only to the states of a definite number of particles. For example, two independent sources of coherent states which have no definite energy can generate an interference pattern without question.

Dirac's argument, however, may seem to apply at least to the particle-number Fock states that do have a definite energy value. Another reason why Fock states seem incapable of interfering is that they do not have a well-defined relative phase. And the last reason—a direct calculation of the first-order correlation function for two Fock states does not reveal any interference properties. Unfortunately, these three very attractive arguments fail.

The first beautiful experimental example of the Fock state interference has been accomplished with Bose-Einstein condensates which can be thought of as particle-number states. In the double-condensate experiment by Andrews *et al.* [3], the authors proved the existence of interference fringes in the measurement of positions of condensate atoms. Results of a similar experiment have been recently reported in Ref. [4]. How is it possible?

The reason is that the first-order correlation function is attributed to an average (over many realizations) density of particles, while in the experiments [3,4] we deal with the results of a single measurement.

Quantum mechanics cannot predict the exact result of a single measurement—it only predicts average values of certain observables or a probability of a definite result of a single experiment. But since we are dealing with a huge number of particles, how about analyzing the many-particle probability distribution to predict a typical particle density profile in a single experiment [5]? Apparently considering only the first-order correlation function is not enough to guess the typical density shape and one needs to take into account higher-order correlation functions.

This issue was first addressed in beautiful work of Javanainen and Yoo [6], where the authors apply numerical analysis to studies of the structure of many-particle probability density distributions. In this and the subsequent numerical simulation [7], by exploiting the laws of quantum mechanics the authors show that two Fock states can indeed reveal an interference pattern in a single measurement of particle positions. Obviously, after averaging out over many realizations of the numerical experiment the interference effects disappear as expected.

In this paper, we analytically study the high-order correlation functions to show directly from their mathematical structure the existence of interference effects in a single interference measurement of two Fock states. In our analysis, we do not use any approximations, as did the previous authors who attempted to prove this result analytically [8,9], assuming orthogonality of the phase states. This approximation is questionable when the number of particles measured is of the order of the total number of particles of the system. Another interesting method of approximating the problem has been shown in [10]. We also show an unintuitive property of the noise present in the interference pattern. The noise level turns out to be slightly lower than in the case of multiple drawn positions with a probability distribution equal to the interference pattern. We have shown in a numerical test that the difference is very small and probably out of reach of any experimental observation. Our result, however, gives an interesting insight into the structure of the high-order correlation function.

In Sec. II we introduce a simple description of the measurement of positions of particles occupying two interfering modes; in Sec. III we discuss the structure of the arbitrary-order correlation function for a pair of Fock states and prove analytically the presence of interference patterns. Finally, Sec. IV concludes our paper.

## II. MODEL OF DETECTION OF PARTICLE POSITIONS

Consider a set of $d$ identical ideal detectors capable of counting particles. Let the surface $L$ of the $i$th detector placed at the position $x_i$ be described by the characteristic function $\chi(x-x_i)$ and the annihilation operator $\hat{A}_i$ associated with the mode $L^{-1/2}\chi(x-x_i)$. Let us assume that the detectors are spatially separated, i.e., $\int\chi(x-x_i)\chi(x-x_j)dx=L\delta_{ij}$.

We calculate an average product of the particle counts from all $d$ detectors [11]:

$$I(x_1, \ldots ,x_d) = \langle \hat{A}_1^\dagger \hat{A}_1 \cdots \hat{A}_d^\dagger \hat{A}_d \rangle. \tag{1}$$

The set of particles measured by the detectors is described by the field operator $\hat{\Psi}(x)$. We assume that the occupied modes are slowly varying in comparison to the size $L$ of the detectors:

$$\hat{A}_i = \int \frac{1}{\sqrt{L}}\chi(x-x_i)\hat{\Psi}(x)dx \approx \sqrt{L}\hat{\Psi}(x_i). \tag{2}$$

From the above formula follows a connection between the average product of detector counts and the $d$-order correlation function:

$$I(x_1, \ldots ,x_d) = L^d\langle \hat{\Psi}^\dagger(x_1)\cdots\hat{\Psi}^\dagger(x_d)\hat{\Psi}(x_1)\cdots\hat{\Psi}(x_d)\rangle. \tag{3}$$

If we assume that the detectors' size $L$ is so small that each of them detects, on average, much less than a single particle, then the average product of the particle counts $I(x_1,\ldots,x_d)$ can be identified with a probability of detection of exactly one particle by each detector. Thus the probability density $\varrho$ of localizing the first particle at the position $x_1$, the second particle at $x_2$, etc., equals

$$\varrho(x_1, \ldots ,x_d) = \frac{(N-d)!}{N!}\langle \hat{\Psi}^\dagger(x_1)\cdots\hat{\Psi}^\dagger(x_d)\hat{\Psi}(x_1)\cdots\hat{\Psi}(x_d)\rangle, \tag{4}$$

where $N$ is the total number of particles. Let us note that the probability density (4) is defined only for states of a definite number of particles. This approach allows one to interpret the physical meaning of the correlation function of order $d$ in two ways. On the one hand, it is proportional to the average product of particle counts of $d$ detectors; on the other hand, it is related to the probability density of localizing exactly one particle by each of $d$ very small detectors in a single measurement. Therefore according to Born's probabilistic interpretation of quantum mechanics a result of such an experiment—a set of measured positions $x_1,\ldots,x_d$—can be treated as a result of a single drawing from the probability density $\varrho(x_1,\ldots,x_d)$.

As long as the detectors are spatially separated, an ordering of the field operators in the expressions (3) and (4) is defined up to the commutation relation $[\hat{\Psi}(x),\hat{\Psi}^\dagger(y)]=\delta(x-y)$. If one wants to continuously extend the expressions to the case of $x_i=x_j$ for $i\neq j$ then the field operators must be ordered normally.

## III. TWO INTERFERING FOCK STATES

Consider a two-mode quantum state $|n,N-n\rangle$ with the first mode defined by an arbitrary function $u(x)$ and the second orthogonal mode by $w(x)$ for $x\in[0,1]$. From the expression (4) we calculate the probability distribution of localizing all the $N$ particles at positions $x_1,x_2,\ldots,x_N$:

$$\varrho_{|n,N-n\rangle}(x_1, \ldots ,x_N) = \binom{N}{n}^{-1}\left|\sum_{\mathcal{P}} u(x_{\mathcal{P}(1)})\cdots u(x_{\mathcal{P}(n)})w(x_{\mathcal{P}(n+1)})\cdots w(x_{\mathcal{P}(N)})\right|^2, \tag{5}$$

where we sum up over permutations $\mathcal{P}$ of an $N$-element set excluding the nontrivial permutations acting separately on the first $n$ elements of the set and the last $N-n$ elements. We will consider the case $N=2n$, when exactly $n$ particles occupy each mode. Using the formula (4) we find that the probability density of detecting $d$ of $2n$ particles at the positions $x_1,x_2,\ldots,x_d$ can be expressed with the probability densities for the asymmetric states (5),

$$\varrho_{|n,n\rangle}(x_1, \ldots ,x_d) = \binom{2n}{n}^{-1}\sum_{j=1}^{d}\Theta(n-j)\Theta(n-d+j) \times\binom{2n-d}{n-d+j}\binom{d}{j}\varrho_{|j,d-j\rangle}(x_1, \ldots ,x_d), \tag{6}$$

where $\Theta(x)$ is the Heaviside theta function. The binomial coefficients $\binom{n}{k}=\frac{n!}{k!(n-k)!}$ showing up in the above expression are for $n\gg 1$ bell-shaped functions of $k$ centered around $k=n/2$ and with a dispersion equal to $\sqrt{n}/2$. We see that the coefficients $\binom{2n-d}{n-d+j}$ and $\binom{d}{j}$ attain their maxima for the same value $j=d/2$, but they are characterized by the different dispersions of the variable $j$: $\sqrt{2n-d}/2$ and $\sqrt{d}/2$, respectively.

### A. Detecting only a small fraction of particles

Consider a special case of the probability density (6), with only a small fraction of all particles being measured, $d\ll n$. In this case the distribution $\binom{2n-d}{n-d+j}$ is much wider than $\binom{d}{j}$ and we can replace the former with its maximum value. In this case also the Heaviside thetas are equal to unity and we can skip them. As a result the expression (6) can be written in the following form:

$$\varrho_{|n,n\rangle}(x_1, \ldots ,x_d) \overset{d\ll n}{\approx} \sum_{j=1}^{d} 2^{-d}\binom{d}{j}\varrho_{|j,d-j\rangle}(x_1, \ldots ,x_d) = \int_{-\pi}^{\pi}\frac{d\phi}{2\pi}\prod_{i=1}^{d}\frac{1}{2}|u(x_i)+e^{i\phi}w(x_i)|^2. \tag{7}$$

We have managed to express the low-order correlation function for the highly occupied state $|n,n\rangle$ in the elegant form of an integral over some positive expression. A similar result has been shown in Refs. [8,9], but the authors omit the fact that they actually prove it only for $d\ll n$ because of the limited validity of the approximations used. These approximations are highly questionable when the number of particles measured is of the order of the total number of particles $d\sim 2n$; therefore we are going to prove all the properties of the high-order correlation functions with no approximations whatsoever.

It turns out, that Eq. (7) tells a lot about the result of a single measurement of the positions of $d$ particles. As we have already pointed out, according to Born the result of such a measurement—the set of measured positions $x_1, \ldots, x_d$—corresponds to a result of a single drawing from the probability density $\varrho_{|n,n\rangle}$. Let us try to predict the result of such a drawing using the following simple lemma based on Bayes' theorem.

*Lemma*. If a $d$-dimensional probability density $\varrho$ can be represented in the form $\varrho(x_1, \ldots, x_d) = \int d\xi p(\xi) q(x_1, \ldots, x_d|\xi)$, where $p$ is a one-dimensional probability density and $q$ is a $d$-dimensional conditional probability distribution (likelihood), then drawing a set of random variables $(x_1, \ldots, x_d)$ with the probability $\varrho$ is equivalent to drawing a random variable $\xi$ with the density $p$, and then drawing the set of random variables $(x_1, \ldots, x_d)$ with the density $q$ for the chosen $\xi$.

*Proof.* The equivalence of both densities can be shown by proving the equality of arbitrary moments of the distributions. We will use an elementary theorem about changing the order of integrals. An arbitrary moment for the second distribution reads

$$\int d\xi\, p(\xi) \int dx_1 \cdots dx_d x_1^{k_1} \cdots x_d^{k_d} q(x_1, \ldots, x_d|\xi)$$
$$= \int dx_1 \cdots dx_d x_1^{k_1} \cdots x_d^{k_d} \int d\xi p(\xi) q(x_1, \ldots, x_d|\xi),$$

and it is equal to the same moment for the distribution $\varrho$. As we know, the values of all the moments uniquely determine the probability distribution.

It follows that the result of a single draw from the probability density (7) can be achieved by a preliminary draw of the parameter $\phi$ with a flat distribution, and then by drawing positions of particles according to the separable density $\prod_{i=1}^{d} \frac{1}{2}|u(x_i)+e^{i\phi}w(x_i)|^2$. The second draw yields positions centered around maxima of the one-dimensional function $\frac{1}{2}|u(x)+e^{i\phi}w(x)|^2$. If we assume $u(x)=w^*(x)=e^{i\pi x}$, $x \in [0,1]$, then every single measurement reveals interference fringes with maxima located randomly each time somewhere else. The meaning of the interference fringes can be made more precise in the following way. Suppose that the whole space of possible particle positions is divided into $D$ small areas of equal length, and we examine how many of the $d$ particles enter each of these areas in a single measurement [12]. Each area is tightly covered by a set of small detectors constituting, so to say, a single superdetector. We look at the histograms of the count statistics of the single measurement—if the sizes of the considered areas are such that each of them swallows on average a large number of particles, then each histogram should reproduce the function $\frac{1}{2}|u(x)+e^{i\phi}w(x)|^2$ for some $\phi$.

### B. Measurement of positions of all particles

We have just shown that for the single measurement of the relatively small number $d$ of particles belonging to the state $|n,n\rangle$ one observes interference fringes. Therefore it is natural to ask about the result of a similar measurement of all the $2n$ particles. Below we show that as the number of particles being measured gets larger, fringes of even higher quality are observed. This agrees with the numerical test [7] and the methodology of the experiment [12].

Our proof is based on the second part of Eq. (7) for $d=2n$:

$$\sum_{j=1}^{2n} 2^{-2n} \binom{2n}{j} \varrho_{|j,2n-j\rangle}(x_1, \ldots, x_{2n})$$
$$= \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \prod_{i=1}^{2n} \frac{1}{2} |u(x_i) + e^{i\phi} w(x_i)|^2. \tag{8}$$

According to our lemma, drawing $2n$ positions described by the right-hand side probability distribution can be achieved by drawing the parameter $\phi$, and then drawing positions with the conditional probability distribution $\prod_{i=1}^{2n} \frac{1}{2}|u(x_i) + e^{i\phi}w(x_i)|^2$. However, according to the expression (8), another equivalent method of drawing exists and is based on drawing first the parameter $j$ described by the distribution $2^{-2n}\binom{2n}{j}$ and then drawing positions of the particles from the probability density $\varrho_{|j,2n-j\rangle}(x_1, \ldots, x_{2n})$ given by the analytic formula (5). Obviously, both methods of drawing lead to the same result, which, as we know, reveals interference patterns of known shape. Equation (8) indicates that the results of drawing of positions with the probability distribution $\varrho_{|j,2n-j\rangle}(x_1, \ldots, x_{2n})$ for the parameter $j$ differing from $n$ by not more than a few dispersion lengths $\sqrt{n/2}$ must reveal the interference effects every time. Independently of the method of drawing, each random histogram will vary from the ideal shape $\frac{1}{2}|u(x)+e^{i\phi}w(x)|^2$ because of statistical fluctuations. Let us introduce the following measure of these fluctuations defined for an arbitrary result of the single drawing. Let the number of counts of the $i$th superdetector placed at $x_i$ be denoted with $n_i$. For the histogram of results $\{n_i\}$ we define the following quantity:

$$\chi^2 = \inf_{\phi} \sum_{i=1}^{D} \left( n_i - \frac{n}{D} |u(x_i) + e^{i\phi} w(x_i)|^2 \right)^2, \tag{9}$$

where $D$ is the number of superdetectors. The above expression averaged out over many realizations $\overline{\chi^2}$ we will call noise. This noise depends only on the number of superdetectors and the probability distribution $\varrho$, or equivalently on the quantum state $|\Psi\rangle$ and the parameter $D$, which we denote as $\overline{\chi^2}(|\Psi\rangle, D)$. Therefore the better the histograms reproduce the shape $\frac{1}{2}|u(x)+e^{i\phi}w(x)|^2$ (for some $\phi$) the lower the value of noise $\overline{\chi^2}$. From Eq. (8) we get

$$\sum_{j=1}^{2n} 2^{-2n} \binom{2n}{j} \overline{\chi^2}(|j, 2n-j\rangle, D) = \overline{\chi^2}(|2n\rangle_{\phi}, D), \tag{10}$$

where $|2n\rangle_{\phi}$ is the so-called phase state of $2n$ particles occupying the same mode $\frac{1}{\sqrt{2}}[u(x)+e^{i\phi}w(x)]$. We have used the fact that the quantity $\overline{\chi^2}(|2n\rangle_{\phi}, D)$ cannot depend on the selection of $\phi$ and it determines the level of noise for the histogram of particle positions drawn one by one with the probability density $\frac{1}{2}|u(x)+e^{i\phi}w(x)|^2$. Equation (10) indicates that the noise level $\overline{\chi^2}(|2n\rangle_\phi, D)$ is equal to an average noise level for the states $|j,2n-j\rangle$ with weights equal to $2^{-2n}\binom{2n}{j}$.

Unfortunately, the level of noise averaged out over all states $|j,2n-j\rangle$ does not uniquely determine the value of the noise $\overline{\chi^2}(|j,2n-j\rangle, D)$ for a particular $j$. However, we can use the natural assumption that the interference effects disappear for the asymmetric states $|j,2n-j\rangle$. In the extreme but highly improbable example of the state $|2n,0\rangle$ or $|0,2n\rangle$, the interference will obviously be completely absent. To be more specific, we assume that $\overline{\chi^2}(|j,2n-j\rangle, D)$ is a monotonically increasing function of $|j-n|$. According to this assumption and Eq. (10), we anticipate the following inequality to hold:

$$\overline{\chi^2}(|n,n\rangle, D) < \overline{\chi^2}(|2n\rangle_\phi, D), \tag{11}$$

which completes the proof.

We have investigated the validity of this inequality by comparing the noise in numerically drawn histograms for the states $|n,n\rangle$ and $|2n\rangle_\phi$, but the observed difference did not exceed the level of statistical error. This means that the difference between $\overline{\chi^2}(|n,n\rangle, D)$ and $\overline{\chi^2}(|2n\rangle_\phi, D)$ is very small, which reflects the fact that the probability of drawing the highly asymmetric state in (8) is negligible. The nonintuitive inequality (11), although very weak, must be an interesting signature of nontrivial spatial correlations present within the mathematical structure of the Fock states.

Let us take a look again at the structure of the analytic expression (6). We notice that, when the number $d$ of the drawn particles approaches its maximum value $2n$, then the width of the distribution $\binom{2n-d}{n-d+j}$ equal to $\frac{\sqrt{2n-d}}{2}$ rapidly shrinks. It follows that the more particles we measure, the more symmetric states (which more likely contribute to the interference) are being chosen for the drawing of the positions.

Our last conclusion is that, in the limit of $d \gg 1$, the probability distribution $2^{-d}\binom{d}{j}$ from Eq. (7) becomes relatively narrow as $\sqrt{d}/2 \ll d$ and only the states $|j,d-j\rangle$ that are almost symmetric will be chosen for the drawing of the particle positions. Therefore in the large-particle-number limit all quantities that weakly depend on the asymmetry of the state will reproduce the results obtained for the phase states $|d\rangle_\phi$.

We have proven the existence of interference effects by studying the structure of the high-order correlation functions for the Fock states. It is also clear that these effects will disappear after averaging out over many repetitions of the measurement. This result is, however, an immediate consequence of the Bogoliubov method, which assumes *ad hoc* that one can replace the field operator of a single condensate by a classical wave with small quantum corrections: $\hat{\Psi} \approx \sqrt{N}e^{i\phi} + \delta\hat{\Psi}$ and then neglect the latter. Our analysis allows one to attribute the arbitrarily chosen phase $\phi$ in the Bogoliubov method with the parameter $\phi$ from Eq. (7) spontaneously induced in a single measurement. In this interpretation, breaking the phase-space symmetry of the Fock states by using the Bogoliubov method corresponds to replacing the strict expressions given by Eq. (6) with their approximations (7).

## IV. CONCLUSIONS

We have shown that the structure of the arbitrary-order correlation function for a pair of Fock states reveals fringes in an interference experiment. In particular, for the measurement of all the particles, we have drawn our conclusion directly from the strict equality (8). Therefore, without referring to the concept of phase or numerical simulations, we have introduced a simple, analytic argument why in a single measurement of the positions of all particles of two interfering Bose-Einstein condensates interference fringes must occur. We have also shown that the noise level present in the statistics is lower than for a corresponding measurement of phase states, although the effect is probably out of reach of any realistic experiment.

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