Algebraic Quantum Theory

A Brief Guide

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April 6, 2012

Abstract

We present the mathematical setting of an algebraic approach to quantum theory, including the Tomita–Takesaki theory, liouvilleans and quantum dynamical systems, and non-commutative integration up to the Falcone–Takesaki theory.

Contents

1 Algebras and functionals

1.1 C^*-algebras

A C^****-algebra*** [29] is defined as an algebra $\mathcal{A}$ over the field $\mathbb{C}$, equipped with an involution map $^*: \mathcal{A} \to \mathcal{A}$ satisfying

$$(ab)^* = b^*a^*, \quad (a+b)^* = a^* + b^*, \quad (a^*)^* = a, \quad (\lambda a)^* = \lambda^* a^*,$$

where $*: \mathbb{C} \ni \lambda =: (x+iy) \mapsto \lambda^* := (x+iy) \in \mathbb{C}$ for any $x, y \in \mathbb{R}$, and equipped with a norm map $\|\cdot\| : \mathcal{A} \to [0, \infty[$ such that $\mathcal{A}$ is a Banach space with norm $\|\cdot\|$, and any of the equivalent conditions

$$\|a^*a\| = \|a\|^2,$$
$$\|a^*a\| = \|a\|\|a^*\|$$

holds. This condition is assumed in order to guarantee that every $*$-homomorphism of C^*-algebra is continuous in terms of $\|\cdot\|$. Algebras over $\mathbb{C}$ that are equipped only with the map $*$ defined as above (but with no norm) are called $*$***-algebras***.

A $*$***-homomorphism*** of $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ is defined as a map $\varsigma : \mathcal{A} \to \mathcal{B}$ such that

$$\varsigma(\lambda_1 a_1 + \lambda_2 a_2) = \lambda_1 \varsigma(a_1) + \lambda_2 \varsigma(a_2),$$
$$\varsigma(a_1 a_2) = \varsigma(a_1)\varsigma(a_2),$$
$$\varsigma(a^*) = \varsigma(a)^*.$$

Each $*$-homomorphism ς of a C^*-algebra $\mathcal{A}$ is continuous in the norm $\|\cdot\|$ of $\mathcal{A}$, and $\|\varsigma(a)\| \leq \|a\|$ holds for each $a \in \mathcal{A}$. A $*$-homomorphism $\varsigma : \mathcal{A} \to \mathcal{B}$ of C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is called $*$***-isomorphism*** iff

$$0 = \ker(\varsigma) := \varsigma^{-1}(0) := \{a \in \mathcal{A} \mid \varsigma(a) = 0\}.$$

A $*$***-automorphism*** of a C^*-algebra $\mathcal{A}$ is defined as a $*$-isomorphism from $\mathcal{A}$ to $\mathcal{A}$. Every $*$-automorphism α of a C^*-algebra satisfies $\|\alpha(a)\| = \|a\| \; \forall a \in \mathcal{A}$. The space of all $*$-automorphisms of a given C^*-algebra $\mathcal{A}$ is denoted by $\mathrm{Aut}\mathcal{A}$.

An element a of a $*$-algebra $\mathcal{A}$ is called: ***normal*** iff $aa^* = a^*a$; ***self-adjoint*** iff $a = a^*$; ***isometry*** iff $\mathcal{A}$ is unital and $a^*a = \mathbb{I}$; ***unitary*** iff $\mathcal{A}$ is unital and $aa^* = a^*a = \mathbb{I}$; ***projection*** iff $a = a^* = aa =: a^2$; ***positive*** iff $\exists b \in \mathcal{A}$ such that $a = bb^*$; ***invertible*** iff $\mathcal{A}$ is unitary and $\exists b \in \mathcal{A}$ such that $ab = \mathbb{I} = ba$ (this is denoted by $b =: a^{-1}$). From this it follows that (a is unitary)$\Rightarrow$(a is invertible), (a is unitary)$\Rightarrow$(a is isometry)$\Rightarrow$(a is normal), (a is projection)$\Rightarrow$(a is positive)$\Rightarrow$(a is self-adjoint)$\Rightarrow$(a is normal), $(a^{-1})^{-1} = a$, $(ab)^{-1} = b^{-1}a^{-1}$, and $(a^*)^{-1} = (a^{-1})^*$. If a is a positive element of a $*$-algebra $\mathcal{A}$, then it is denoted $a \geq 0$. A partial order relation $\geq$ on elements of $*$-algebra $\mathcal{A}$ is defined by

$$a \geq b \iff (a-b) \geq 0, \text{ for } a, b \in \mathcal{A}.$$

The set of all positive elements of a $*$-algebra $\mathcal{A}$ is called its ***cone*** and denoted $\mathcal{A}^+$.[1] It is convex and closed in norm topology.

The C^*-algebra $\mathcal{A}$ is called to be ***generated*** by a set $\mathcal{D} \subseteq \mathcal{A}$ if $\mathcal{A}$ is equal to completion of the algebra of polynomials of elements of $\mathcal{D}$ provided in terms of the norm $\|\cdot\|$ of $\mathcal{A}$. In particular, if $\mathcal{D} \subseteq \mathcal{A}$ is a C^*-algebra generated by the set $\{a, a^*\}$, where $a, a^* \in \mathcal{A}$, then $(a - \lambda\mathbb{I})^{-1} \subseteq \mathcal{D}$.

[1] By the the Gel'fand–Naĭmark construction of the spectra of a commutative C^*-algebra and a fact

1.2 Functionals

A functional $\omega : \mathcal{A} \to \mathbb{C}$ on a C^*-algebra $\mathcal{A}$ is called: ***linear*** iff $\omega(\lambda_1 A + \lambda_2 B) = \lambda_1\omega(A) + \lambda_2\omega(B)$; ***positive*** iff $\omega(A^*A) \geq 0$; ***faithful*** iff $\omega(A^*A) = 0 \Rightarrow A = 0$; ***tracial*** iff $\omega(AB) = \omega(BA)$; ***normalised*** iff $\omega(\mathbb{I}) = 1$; ***hermitean*** iff $\omega(A^*) = \omega(A)^*$. If ω, ϕ and $(\omega - \phi)$ are positive functionals on $\mathcal{A}$, then one writes $\phi \leq \omega$, and says that ω ***majorises*** ϕ. The hermitean functionals are uniquely determined by their restriction to the self-adjoint elements of $\mathcal{A}$. The norm of a linear functional is defined as

$$\|\omega\| := \sup\{|\omega(A)| \mid \|A\| \leq 1 \; \forall A \in \mathcal{A}\}. \tag{1}$$

Each positive linear functional on a unital C^*-algebra satisfies $\|\omega\| = \omega(\mathbb{I})$. A positive normalised linear functional $\omega : \mathcal{A} \to \mathbb{C}$ on a C^*-algebra $\mathcal{A}$ is called an ***algebraic state*** [56]. Every algebraic state is hermitean, hence it can be completely determined by the values it takes on the self-adjoint elements of the algebra.

The ***Banach dual*** $\mathcal{B}^{\mathbf{B}}$ of the Banach space $\mathcal{B}$ is defined as a space of all linear functionals ϕ on $\mathcal{B}$ that are ***continuous in norm topology***, that is

$$\|A_n - A_m\| \to 0 \;\Rightarrow\; |\phi(A_n) - \phi(A_m)| \to 0 \; \forall \{A_i\} \in \mathcal{B}.$$

The ***weak-* topology*** on $\mathcal{B}^{\mathbf{B}}$ is defined as the weakest topology on $\mathcal{B}^{\mathbf{B}}$ such that the maps $\mathcal{B}^{\mathbf{B}} \ni \omega \mapsto \omega(A) \in \mathbb{C}$ are continuous for every $A \in \mathcal{B}$. The neighbourhoods of the weak-* topology on $\mathcal{B}^{\mathbf{B}}$ have the form

$$N_{\epsilon,\{A_k\}}(\phi) := \{\omega \in \mathcal{B}^{\mathbf{B}} \mid |\omega(A_k) - \phi(A_k)| < \epsilon, \; A_k \in \mathcal{B}\},$$

with $\epsilon > 0$, $k \in \{1, \ldots, m\}$, and arbitrary finite $m \in \mathbb{N}$. This topology is locally convex. The space $\mathcal{B}_*$, defined as such Banach space that satisfies $\mathcal{B}_*^{\mathbf{B}} = \mathcal{B}$, is a subset of the space of a linear functionals on $\mathcal{B}$, and is called a ***predual*** of $\mathcal{B}$.

If $\mathcal{A}$ is a C^*-algebra, then the space $\mathcal{A}^{\mathbf{B}}$ is a Banach space with a norm given by (1). The space of all positive linear functionals on $\mathcal{A}$ is denoted $\mathcal{A}^{\mathbf{B}+}$. The space of all algebraic states on a C^*-algebra $\mathcal{A}$ is given by

$$\mathcal{S}(\mathcal{A}) := \{\omega \in \mathcal{A}^{\mathbf{B}+} \mid \|\omega\| = 1\}.$$

The space of all faithful algebraic states is denoted $\mathcal{S}_0(\mathcal{A})$. If for a given C^*-algebra $\mathcal{A}$ there exists a predual $\mathcal{A}_*$, then $\mathcal{A}$ is called a ***W^*-algebra***, [while the linear functionals in $\mathcal{A}_*$ are called *normal*]. Every normal functional $\omega \in \mathcal{A}_*$ is continuous in ***σ-weak topology***, defined as a locally convex topology generated by the family of semi-norms $\mathcal{A} \ni A \mapsto |\omega(A)| \in \mathbb{C}$.[2] In general, this topology is not first countable. The norm on $\mathcal{A}_*$

that every self-adjoint element x of any C^*-algebra has $\mathrm{sp}(x) \in \mathbb{R}$, the above definition of partial order $x \geq y$ for $x, y \in \mathcal{A}$ on an arbitrary C^*-algebra $\mathcal{A}$ is equal with the partial order defined by

1. $x \geq 0 \iff x = x^*$ and $\mathrm{sp}(x) \subseteq [0, \infty[$,
2. $x \geq y \iff x - y \geq 0$.

Moreover, if the C^*-algebra $\mathcal{A}$ under consideration is $\mathfrak{B}(\mathcal{H})$, then the positivity of $a \in \mathcal{A}$ implies that $\langle \xi, a\xi \rangle = \|b\xi\|^2 \geq 0 \; \forall \xi \in \mathcal{H}$, where $b \in \mathcal{A}$ and $a = bb^*$.

[2] A functional $\omega \in \mathcal{A}^{\mathbf{B}}$ is continuous in σ-weak topology iff $\omega(\sup_\iota A_\iota) = \sup_\iota \omega(A_\iota)$ for every bounded and increasing net $\{A_\iota\} \in \mathcal{A}$ such that $A_\iota = A_\iota^*$.

coincides with the norm on $\mathcal{A}^{\mathbf{B}}$. One defines also $\mathcal{A}_*^+ := \mathcal{A}^{\mathbf{B}+} \cap \mathcal{A}_*$ and $\mathcal{A}_{*1}^+ := \mathcal{S}(\mathcal{A}) \cap \mathcal{A}_*$. The space $\mathcal{S}(\mathcal{A}) \cap \mathcal{A}_*$ of all normal algebraic states is dense in $\mathcal{S}(\mathcal{A})$ in the weak-* topology. By the weak-$*$ compactness theorem [the Banach–Alaoğlu–Bourbaki–Kakutani–...], $\mathcal{S}(\mathcal{A})$ is a convex subset of $\mathcal{A}^{\mathbf{B}}$ which is compact in weak-$*$ topology.

1.3 Representations and the Gel'fand–Naĭmark–Segal construction

A ***representation*** of a C^*-algebra $\mathcal{A}$ on a given Hilbert space $\mathcal{H}$ is defined as a $*$-homomorphism $\pi : \mathcal{A} \to \mathfrak{B}(\mathcal{H})$. It is called: ***faithful*** iff $\ker(\pi) = \{0\}$ (iff $\pi(a) > 0 \; \forall a > 0$); ***non-degenerate*** iff $\{\pi(a)\psi \mid \forall a \in \mathcal{A} \; \forall \psi \in \mathcal{H}\}$ is dense in $\mathcal{H}$. Any representation $\pi : \mathcal{A} \to \mathfrak{B}(\mathcal{H})$ is a faithful representation of an algebra $\mathcal{A}/\ker(\pi) \equiv \mathcal{A}/\pi^{-1}(\{0\})$. Any representation $\pi : \mathcal{A} \to \mathfrak{B}(\mathcal{H})$ of a unital C^*-algebra is non-degenerate iff $\pi(\mathbb{I}) = \mathbb{I}$. The representations $\pi_1 : \mathcal{A} \to \mathfrak{B}(\mathcal{H}_1)$ and $\pi_2 : \mathcal{A} \to \mathfrak{B}(\mathcal{H}_2)$ are called ***equivalent*** or ***unitarily equivalent*** iff $\exists$ a (unitary) isomorphism $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\pi_2(a) = U\pi_1(a)U^{-1} \; \forall a \in \mathcal{A}$. If the representations $\pi_1 : \mathcal{A} \to \mathfrak{B}(\mathcal{H}_1)$ and $\pi_2 : A \to \mathfrak{B}(\mathcal{H}_2)$ are not unitarily equivalent, they are called ***inequivalent*** or ***unitarily inequivalent***.

Every abstract C^*-algebra $\mathcal{A}$ and any algebraic state $\omega \in \mathcal{S}(\mathcal{A})$ uniquely define the Hilbert space by the following Gel'fand–Naĭmark–Segal (GNS) [29, 56] construction. For a C^*-algebra $\mathcal{A}$, elements $A, B \in \mathcal{A}$ and an algebraic state $\omega : \mathcal{A} \to \mathbb{C}$, one defines the scalar form $\langle \cdot, \cdot \rangle_\omega$ on $\mathcal{A}$,

$$\langle A, B \rangle_\omega := \omega(A^* B),$$

and the ***Gel'fand ideal*** $\mathcal{I}_\omega := \{A \in \mathcal{A} : \langle A, A \rangle_\omega = 0\}$, which is a closed left ideal of $\mathcal{A}$. The Hilbert space $\mathcal{H}_\omega$ is obtained by the completion of $\mathcal{A}/\mathcal{I}_\omega$ in the topology generated by $\langle \cdot, \cdot \rangle_\omega$. The dense subspaces of $\mathcal{H}_\omega$ are then the equivalence classes of elements of $\mathcal{A}$ modulo the ideal $\mathcal{I}_\omega$. The form $\langle \cdot, \cdot \rangle_\omega$ is hermitean on $\mathcal{A}$ and it becomes a scalar product $\langle \cdot, \cdot \rangle_\omega$ on $\mathcal{A}/\mathcal{I}_\omega$. The state ω defines a representation π_ω of an abstract C^*-algebra $\mathcal{A}$ in the space $\mathfrak{B}(\mathcal{H}_\omega)$ of all bounded linear operators on $\mathcal{H}_\omega$ by the morphism

$$\eta_\omega : \mathcal{A} \ni A \longmapsto [A]_\omega \in \mathcal{A}/\mathcal{I}_\omega$$

and

$$\pi_\omega(A) : \eta_\omega(B) \longmapsto \eta_\omega(AB).$$

This representation is called the ***Gel'fand–Naĭmark–Segal representation***. This way the algebraic state ω leads to construction of the concrete Hilbert space $\mathcal{H}_\omega$ from an abstract C^*-algebra $\mathcal{A}$ and to representation π_ω of this algebra in the form of the *concrete* C^*-algebra $\pi_\omega(\mathcal{A})$ acting on $\mathcal{H}_\omega$. The GNS representation is faithful and non-degenerate. The algebraic state ω is uniquely represented in terms of $\mathcal{H}_\omega$ by the vector $\eta_\omega(\mathbb{I}) =: \Omega_\omega \in \mathcal{H}_\omega$. This vector is ***cyclic*** for $\pi_\omega(\mathcal{A})$, which means that the set $\{\pi_\omega(A)\Omega_\omega \mid A \in \mathcal{A}\}$ is dense in $\mathcal{H}_\omega$ in norm topology. Hence

$$\forall A \in \mathcal{A} \quad \omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle_\omega . \tag{2}$$

The GNS representation is a unique representation, up to unitary equivalence, such that there exists an element $\Omega_\omega \in \mathcal{H}_\omega$ that is cyclic for $\pi_\omega(\mathcal{A})$, (2) holds, and $\|\Omega_\omega\|^2 = \|\omega\| = 1$.

The cyclic character of Ω_ω enables the approximation $\omega_\xi(A) = \langle\xi, \pi_\omega(A)\xi\rangle_\omega$ for every $\xi \in \mathcal{H}_\omega$ by such $\omega(B^*AB)$ that $B \in \mathcal{A}$ and $\xi \in \mathcal{H}_\omega$ is weak-$*$ densely approximated by $\pi_\omega(B)\Omega_\omega$ (for this reason we will use the notation $\xi = \eta_\omega(B)$). Moreover, every representation of a C^*-algebra can be decomposed into a direct sum of representations that are unitarily equivalent to the GNS representation.

1.4 The von Neumann algebras

The ***commutant*** of a subalgebra $\mathcal{N}$ of an algebra $\mathcal{A}$ is defined as

$$\mathcal{N}^\bullet := \{b \in \mathcal{A} \mid ab = ba \; \forall a \in \mathcal{N}\},$$

while the ***center*** of $\mathcal{N}$ is defined as $\mathfrak{Z}_\mathcal{N} := \mathcal{N} \cap \mathcal{N}^\bullet$. The commutant operation satisfies:

$$\mathcal{N}_1 \subset \mathcal{N}_2 \Rightarrow \mathcal{N}_1{}^\bullet \subset \mathcal{N}_2{}^\bullet, \; \mathcal{N} \subseteq \mathcal{N}^{\bullet\bullet}, \; \mathcal{N}^{\bullet\bullet\bullet} = \mathcal{N}^\bullet.$$

If $\mathcal{N} \subset \mathcal{N}^\bullet$, then $\mathcal{N}$ is a commutative algebra. A subalgebra $\mathcal{N}$ of a unital algebra $\mathcal{A}$ is called: ***irreducible*** iff $\mathcal{N}^\bullet = \{\lambda\mathbb{I}\}$, $\lambda \in \mathbb{C}$; ***reducible*** iff $\neg(\mathcal{N}^\bullet = \{\lambda\mathbb{I}\})$; a ***factor*** iff $\mathfrak{Z}_\mathcal{N} = \{\lambda\mathbb{I}\}$. Every unital commutative algebra is a factor. A unital $*$-subalgebra $\mathcal{N}$ of an algebra $\mathfrak{B}(\mathcal{H})$ is called the ***von Neumann algebra*** [73, 46] iff $\mathcal{N} = \mathcal{N}^{\bullet\bullet}$. In particular, $\mathfrak{B}(\mathcal{H})$ is the von Neumann algebra.

Every pair of C^*-algebra $\mathcal{A}$ and an algebraic state $\omega \in \mathcal{S}(\mathcal{A})$ generates the von Neumann algebra $\mathcal{N}$ by $\mathcal{N} := (\pi_\omega(\mathcal{A}))^{\bullet\bullet}$, called the ***enveloping von Neumann algebra***. The representation $\pi : \mathcal{N} \to \mathfrak{B}(\mathcal{H})$ of a W^*-algebra $\mathcal{N}$ is called ***normal*** iff it is continuous in σ-weak topology. An example of a normal representation is given by the GNS representation associated with a normal algebraic state ω on a W^*-algebra. The image $\pi(\mathcal{N})$ of representation π of a W^*-algebra $\mathcal{N}$ is a von Neumann algebra iff π is normal and non-degenerate.

The representation π of a unital C^*-algebra $\mathcal{A}$ is called ***irreducible*** iff $\pi(\mathcal{A})$ is irreducible, hence, iff $\pi(\mathcal{A})^\bullet = \{\lambda\mathbb{I}\}$. This is equivalent to the condition: (every $\mathcal{H} \ni \xi \neq 0$ is cyclic for $\pi(\mathcal{A})$) or ($\pi(\mathcal{A}) = \{0\}$ and $\mathcal{H} = \mathbb{C}$). The GNS representation π_ω and the algebra $\pi_\omega(\mathcal{A})$ are irreducible iff ω is pure, that is, iff ω is an extremal point of the convex set $\mathcal{S}(\mathcal{A})$. If $\pi(\mathcal{A})$ is reducible, then every $A \in (\pi(\mathcal{A}))^\bullet$ such that $A = A^*$ and $A \neq \lambda\mathbb{I}$ provides a decomposition of $\mathcal{H}$, given by unitary isomorphism (coming from the spectral theorem) of $\mathcal{H}$ into a sum of subspaces of $\mathcal{H}$ corresponding to the spectrum of A, and called a ***superselection*** [76]. The superselection sectors (subspaces) correspond to unitarily inequivalent representations of $\mathcal{A}$ [?]. The linear combination of vectors which belong to different superselection subspaces of $\mathcal{H}$ are not cyclic with respect to [$\pi_\omega(\mathcal{A})/\pi(\mathcal{A})$?], hence they do not define any spectral measure and cannot be equipped with a quantitative meaning via spectral theory.

1.5 Weights

A ***weight*** [7, 8, 52, 53] on a C^*-algebra $\mathcal{A}$ is defined as a linear map $\omega : \mathcal{A}^+ \to [0, +\infty]$. The definition of a weight can be extended by linearity to the subset

$$\mathfrak{m}_\omega := \{a^*b \mid a, b \in \mathcal{A}, \; \omega(a^*a) < \infty, \; \omega(b^*b) < \infty\} \subseteq \mathcal{A}. \tag{3}$$

A weight is called: ***normalised*** iff $\omega(\mathbb{I}) = 1$; ***faithful*** iff $\omega(a) = 0 \Rightarrow a = 0$; ***finite*** iff $\omega(a) < \infty\ \forall a \in \mathcal{A}^+$; ***semi-finite*** iff

$$\forall a \in \mathcal{A}^+\ \exists b \in \mathcal{A}^+\ (a \geq b,\ b \neq 0,\ \omega(b) < \infty);$$

normal iff for each directed filter $\mathfrak{F} \subset \mathcal{A}^+$ with the upper bound $\sup \mathfrak{F}$ it holds that $\omega(\sup \mathfrak{F}) = \sup_{a \in \mathfrak{F}} \omega(a)$. A space of all semi-finite normal weights on a C^*-algebra $\mathcal{A}$ is denoted $\mathcal{W}(\mathcal{A})$, while the space of all semi-finite faithful normal weights on $\mathcal{A}$ is denoted $\mathcal{W}_0(\mathcal{A})$.

There exists an analogue of the Gel'fand–Naĭmark–Segal representation theorem for weights. Let $\mathcal{A}$ be a C^*-algebra, let ω be a weight on $\mathcal{A}$, and

$$\mathfrak{n}_\omega := \{a \in \mathcal{A} \mid \omega(a^* a) < \infty\}.$$

Then there exists the Hilbert space $\mathcal{H}_\omega$, defined as the completion of $\mathfrak{n}_\omega$ in the norm generated by the scalar product $\mathfrak{n}_\omega \times \mathfrak{n}_\omega \ni (a, b) \mapsto \omega(a^* b) \in \mathbb{C}$, and there exist maps $\eta_\omega : \mathfrak{n}_\omega \to \mathcal{H}_\omega$ and $\pi_\omega : \mathcal{A} \to \mathfrak{B}(\mathcal{H}_\omega)$ such that η_ω is linear, $\mathrm{ran}(\eta_\omega)$ is dense in $\mathcal{H}_\omega$, π_ω is a representation of $\mathcal{A}$ and

$$(\eta_\omega(b), \pi_\omega(a)\eta_\omega(c)) = \omega(b^* a c)\ \ \forall a \in \mathcal{A}\ \ \forall b, c \in \mathfrak{n}_\omega.$$

1.6 Cyclic and separating vectors

For $\omega \in \mathcal{N}_*$ define

$$\mathrm{supp}(\omega) := \inf\{P \in \mathcal{P}(\mathcal{N}) \mid \omega(P) = 1\},$$
$$\mathrm{supp}_{\mathfrak{Z}}(\omega) := \inf\{P \in \mathcal{P}(\mathcal{N}) \cap \mathfrak{Z}_{\mathcal{N}} \mid \omega(P) = 1\}$$

called, respectively, the ***support*** and the ***central support*** of ω. An algebraic state ω is faithful iff $\mathrm{supp}(\omega) = \mathbb{I}$, while π_ω is faithful iff $\mathrm{supp}_{\mathfrak{Z}}(\omega) = \mathbb{I}$ (in such case ω is called ***centrally faithful***). If $\omega(A) = \langle \Omega_\omega, A \Omega_\omega \rangle\ \forall A \in \mathcal{N}$ then Ω_ω is cyclic iff $\mathrm{supp}(\omega)^\bullet = \mathbb{I}$, hence ω is faithful for $\mathcal{N}$ iff Ω_ω is cyclic for $\mathcal{N}^\bullet$.

An element $\xi \in \mathcal{H}$ is called ***separating*** for a C^*-algebra $\mathcal{A} \subset \mathfrak{B}(\mathcal{H})$ iff $A\xi = 0 \Rightarrow A = 0\ \forall A \in \mathcal{A}$. An element $\xi \in \mathcal{H}$ is separating for $\mathcal{A}$ iff $\omega_\xi(\cdot) := \langle \xi, \cdot\ \xi \rangle$ is faithful for $\mathcal{A}$. From this it follows that (ξ is cyclic for $\mathcal{A}$) $\iff$ (ξ is separating for $\mathcal{A}^\bullet$) $\forall \mathcal{A} \subset \mathfrak{B}(\mathcal{H})\ \forall \xi \in \mathcal{H}$. If ω is a faithful algebraic state on an abstract C^*-algebra $\mathcal{A}$, then its cyclic GNS representative $\Omega_\omega \in \mathcal{H}_\omega$ is also separating for $\pi_\omega(\mathcal{A})$. Hence, $\pi_\omega(A)\Omega_\omega \neq 0\ \forall \pi_\omega(A) \neq 0$, because the GNS representation is faithful. If ω is a faithful normal algebraic state on a von Neumann algebra $\mathcal{N}$, then $\pi_\omega(\mathcal{N})^{\bullet\bullet} = \pi_\omega(\mathcal{N}) \cong \mathcal{N}$. Moreover, the following conditions are equivalent:

1. there exists a faithful normal algebraic state on $\mathcal{N}$,
2. $\mathcal{N}$ is isomorphic to a von Neumann algebra possessing a cyclic and separating vector,
3. every family $\{P_n\} \in \mathcal{N}$ of mutually orthogonal ($P_i P_j = 0$ for $i \neq j$) projections is countable.

If any of these conditions is satisfied, then $\mathcal{N}$ is called ***countably finite*** or ***separable***. In particular, $\mathfrak{B}(\mathcal{H})$ is separable iff $\mathcal{H}$ is separable. Note that, while only separable von Neumann algebras admit faithful normal algebraic states, every von Neumann algebra admits a faithful normal semi-finite weight.

1.7 Normal extensions

If $\mathcal{A}$ is a C^*-algebra, $\omega \in \mathcal{S}(\mathcal{A})$ and $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is the associated GNS representation, then $\tilde{\omega}(\cdot) := \langle \Omega_\omega, \cdot \Omega_\omega \rangle$ defines a centrally faithful normal state on the von Neumann algebra $\pi_\omega(\mathcal{A})^{\bullet\bullet}$ that is a unique normal extension of the algebraic state $\pi_\omega(\mathcal{A}) \ni A \mapsto \omega(A) \in \mathbb{C}$ to $\pi_\omega(\mathcal{A})^{\bullet\bullet} \ni A \mapsto \omega(A) \in \mathbb{C}$ (an ***extension*** means here that $\tilde{\omega}|_{\pi_\omega(\mathcal{A})} = \omega$ and $\pi_\omega(\mathcal{A})$ is dense in $\pi_\omega(\mathcal{A})^{\bullet\bullet}$). By an abuse of language, $\tilde{\omega}$ is called a ***normal extension*** of ω to $\pi_\omega(\mathcal{A})^{\bullet\bullet}$.

2 Modular theory

2.1 The Tomita–Takesaki theory

For any given C^*-algebra $\mathcal{A}$, any given one-parameter group of automorphisms[3] $\alpha : \mathbb{R} \in t \mapsto \alpha_t := \{\mathcal{A} \ni A \mapsto A(t) = \alpha_t(A) \in \mathcal{A}\}$, and any $\mathbb{R} \ni \beta > 0$, the algebraic state ω is called ***Kubo–Martin–Schwinger*** (KMS) ***for*** α_t ***and*** β [42, 44, 31, 41] iff

$$\omega(\alpha_t(A)B) = \omega(B\alpha_{t+i\beta}(A)) \tag{4}$$

holds and the maps $\mathbb{R} \ni t \mapsto \omega(\alpha_t(A)B) \in \mathbb{C}$ and $\mathbb{R} \ni t \mapsto \omega(A\alpha_t(B)) \in \mathbb{C}$, can be continued for $t \in \mathbb{C}$ in such way that they are analytic, respectively, for $\mathrm{im}(t) \in]-\beta, 0[$ and $\mathrm{im}(t) \in]0, \beta[$, as well as bounded and continuous in the closure of these domains.

Let $\mathcal{N}$ be the von Neumann algebra acting on the Hilbert space $\mathcal{H}$, and let $\Omega \in \mathcal{H}$ be cyclic ($\mathcal{N}\Omega = \mathcal{H}$) and separating ($A\Omega = 0 \Rightarrow A = 0\ \forall A \in \mathcal{N}$) for $\mathcal{N}$. Define a conjugate linear operator R_Ω, acting on a dense subspace of $\mathcal{H}$ generated by action of the von Neumann algebra $\mathcal{N}$ on Ω, by

$$R_\Omega A\Omega := A^*\Omega.$$

Its closure (denoted with the abuse of notation by the same letter) has a unique polar decomposition $R_\Omega = J_\Omega \Delta_\Omega^{1/2}$ with positive self-adjoint operator Δ_Ω and anti-unitary operator J_Ω. These operators satisfy

$$\begin{gathered} R_\Omega = R_\Omega^{-1} = \Delta_\Omega^{-1/2} J_\Omega^*, \quad R_\Omega^* = J\Delta_\Omega^{-1/2}, \quad \Delta_\Omega^{-1} = R_\Omega R_\Omega^*, \\ \Delta_\Omega = R_\Omega^* R_\Omega, \quad \Delta_\Omega \Omega = \Omega, \quad J_\Omega \Omega = \Omega, \quad J_\Omega \Delta_\Omega J_\Omega = \Delta_\Omega^{-1}, \\ J_\Omega^2 = \mathbb{I}, \quad J_\Omega^* = J_\Omega, \quad \langle JA, JB \rangle = \langle B, A \rangle\ \forall A, B \in \mathcal{N}. \end{gathered}$$

The Tomita theorem [68, 69] states that for every von Neumann algebra $\mathcal{N}$, acting on the Hilbert space $\mathcal{H}$ equipped with a vector $\Omega \in \mathcal{H}$ which is cyclic and separating for $\mathcal{N}$, there exists a *unique* strongly continuous group $\sigma_t^\omega : \mathcal{N} \to \mathcal{N}$, $t \in \mathbb{R}$, of unitary automorphisms of $\mathcal{N}$ such that

[3] If the C^*-algebra is not separable, then α_t is additionally assumed to be [strongly continuous] [23].

1. $\sigma_t^\omega : A \mapsto \sigma_t^\omega(A) := \Delta_\omega^{it} A \Delta_\omega^{-it} = U_\omega(t) A U_\omega^*(t)$,

and a unique anti-linear operator J_ω such that

2. $J_\omega^2 = \mathbb{I}$, $J_\omega \mathcal{N} J_\omega = \mathcal{N}^\bullet$,

3. $U_\omega(-\frac{i}{2}) A\Omega = J_\omega A^* \Omega$.

The Takesaki [63] theorem states moreover that

4. $\omega(A) := \langle \Omega, A\Omega \rangle \ \forall A \in \mathcal{N}$ is a unique algebraic state that is KMS for σ^ω and $\beta = 1$.

These two theorem are sometimes joined together under the label of Tomita–Takesaki theorem.

The group $U_\omega(t) = \Delta_\omega^{it}$ of $*$-automorphims of $\mathcal{N}$ is called a group of ***modular automorphisms***, Δ_ω is called a ***modular operator***, while J_ω is called a ***modular conjugation*** and it defines the anti-linear $*$-isomorphism $j_\omega : \mathcal{N} \ni A \mapsto J_\omega A J_\omega \in \mathcal{N}^\bullet$. The ***modular hamiltonian*** K_ω, defined by

$$e^{-K_\omega} := \Delta_\omega,$$

satisfies $K_\omega^* = K_\omega$ and $K_\omega \Omega = 0$. The unitary operators Δ_ω^{it} leave Ω invariant: $U_\omega(t)\Omega = \Delta_\omega^{it}\Omega = \Omega$. By definition,

$$U_\omega(t) = (e^{-K_\omega})^{it} = e^{-iK_\omega t}. \tag{5}$$

The modular automorphism $\sigma_t^\omega A = U_\omega(t) A U_\omega^*(t)$ is unitary, hence

$$\omega(A) = \omega(\sigma_t^\omega(A)).$$

The algebraic state ω on a C^*-algebra $\mathcal{A}$ is called ***modular*** iff its cyclic GNS representative $\Omega_\omega \in \mathcal{H}_\omega$ is a separating vector for the von Neumann algebra $\pi_\omega(\mathcal{A})^{\bullet\bullet}$, that is, iff ω naturally extends to a faithful normal state on $\pi_\omega(\mathcal{A})^{\bullet\bullet}$. The GNS representative Ω_ω of every faithful algebraic state ω on a C^*-algebra $\mathcal{A}$ is cyclic and separating for $\pi_\omega(\mathcal{A})$ and $\pi_\omega(\mathcal{A})^{\bullet\bullet}$ on $\mathcal{H}_\omega$, hence faithful algebraic states are modular. In consequence, every pair of a C^*-algebra $\mathcal{A}$ and a faithful algebraic state $\omega \in \mathcal{S}_0(\mathcal{A})$ generates a unique Hilbert space $\mathcal{H}_\omega$ equipped with a unique unitary modular automorphism σ_t^ω of the von Neumann algebra $\pi_\omega(\mathcal{A})^{\bullet\bullet}$.[4] Moreover, $\omega \in \mathcal{S}_0(\mathcal{A})$ is always KMS for $\pi_\omega(\mathcal{A})^{\bullet\bullet}$ and $\beta = 1$.

Due to the role played by the KMS condition in the modular theory, it is worth to consider its properties:

1. if ω is KMS for α_t then $\omega(\alpha_t(A)) = \omega(A) \ \forall A \in \mathcal{A}$,

2. if ω is KMS for α_t then $\alpha_t(A) = A \ \forall A \in \mathrm{supp}_{\mathfrak{Z}_\mathcal{A}}(\omega)\mathfrak{Z}_\mathcal{A}$,

3. if ω is KMS for α_t and $\beta \neq 0$, then it is also KMS for $\alpha_{\lambda t}$ and $\frac{\beta}{\lambda}$, where $\mathbb{R} \ni \lambda \neq 0$,

4. ω is KMS for $\beta = 0 \iff \omega(xy) = \omega(yx) \ \forall x, y \in \mathcal{A}$,

[4] Note that this result holds for every $\omega \in \mathcal{S}_0(\mathcal{A})$, hence it is independent reducibility or irreducibility of GNS representation π_ω.

5. ω is KMS and faithful $\iff$ π_ω is faithful,

6. if ω is KMS on $\mathcal{A}$ with respect to α_t and β, and if $\mathcal{H}_\omega$ is separable, then the normal extension $\tilde{\omega}$ of ω is KMS on $\pi_\omega(\mathcal{A})^{\bullet\bullet}$ with respect to $\tilde{\alpha}_t$ and β, where $\widetilde{\alpha}_t \in \mathrm{Aut}(\pi_\omega(\mathcal{A})^{\bullet\bullet})$ is a ***normal extension*** of α_t, defined by [...].

7. if $\mathcal{A}$ is a W^*-algebra, then ω is KMS for α_t and β iff ω is normal and there exists a σ-weakly dense $*$-subalgebra $\mathcal{D} \subseteq \mathcal{A}$, analytic and invariant with respect to the action of α_t, such that
$$\omega(A\alpha_{i\beta}(B)) = \omega(BA)\ \forall A, B \in \mathcal{D}.$$
If $\mathcal{A}$ is a C^*-algebra, then ω is KMS for α_t and β iff there exists a norm dense $*$-subalgebra $\mathcal{D} \subseteq \mathcal{A}$ as above.

The set $\mathcal{S}^{\beta,\alpha}_{\mathrm{KMS}}(\mathcal{A}) \subseteq \mathcal{S}(\mathcal{A})$ of all KMS states on a C^*-algebra $\mathcal{A}$ for a fixed value of β and fixed automorphism α is a convex space which is compact in weak-$*$ topology. Moreover, it is a *simplex*, what means that every element of the space $\mathcal{S}^{\beta,\alpha}_{\mathrm{KMS}}(\mathcal{A})$ can be uniquely decomposed as a convex combination of the extremal elements of this space [63]. The KMS state ω_β is an extremal element of $\mathcal{S}^{\beta,\alpha}_{\mathrm{KMS}}(\mathcal{A})$ iff $\pi_{\omega_\beta}(\mathcal{A})$ is a factor. If ω_β is not extremal, then the spectrum of the center $\mathfrak{Z}_{\pi_{\omega_\beta}}(\mathcal{A})$ is labelled by the extremal KMS states which provide a unique decomposition of ω_β. If there exists a unique KMS state for a given β and α, $\mathcal{S}^{\beta,\alpha}_{\mathrm{KMS}}(\mathcal{A}) = \{\omega\}$, then $\mathfrak{Z}_{\pi_\omega}(\mathcal{A}) \cong \mathbb{C}\mathbb{I}$. If von Neumann algebra $\mathcal{N}$ is finite, then $\mathcal{S}^{\beta,\alpha}_{\mathrm{KMS}}(\mathcal{N}) = \{\omega\}$. When considered on the GNS representation of $\mathcal{N}$, this ω takes the form [ale tylko dla okreslonych sytuacji, gdy $e^{-\beta H}$ ma w ogole sens. podac precyzyjnie jakie to sytuacje.]
$$\omega(\cdot) = \frac{\mathrm{tr}(\rho\cdot)}{\mathrm{tr}(\rho)} : \pi_\omega(\mathcal{N}) \to \mathbb{C},$$
with $\rho \in \mathfrak{B}_1(\mathcal{H})$ according to (9), and satisfying $\rho = e^{-\beta H}$, where H is a self-adjoint operator generating the one-parameter group of unitary operators $U(t)$ corresponding uniquely to α_t by the (10). If ω_{β_1} and ω_{β_2} are KMS on $\mathcal{A}$ with the corresponding generators H_1 and H_2 of their respective automorphisms α_{1t} and α_{2t}, then $\omega_{\beta_1} \otimes \omega_{\beta_2}$ is also KMS, with a generator of corresponding automorphism given by $\beta_1 H_1 + \beta_2 H_2$.[5]

There exists an analogue of the Tomita–Takesaki theory for weights. If ω is a faithful normal semi-finite weight on a von Neumann algebra $\mathcal{N}$, then there exists a closeable operator
$$R_\omega : \eta_\omega(\mathfrak{m}_\omega) \ni \eta_\omega(A) \mapsto \eta_\omega(A^*) \in \mathcal{H}_\omega,$$
where $\mathfrak{m}_\omega \subset \mathcal{N}$ is defined by (3). Its closure, denoted with the abuse of notation by the same symbol, has a polar decomposition $R_\omega = J_\omega \Delta_\omega^{1/2}$ that satisfies
$$\Delta_\omega = R^*_\omega R_\omega,\ \Delta_\omega^{-1} = R_\omega R^*_\omega,\ J_\omega = J^*_\omega,\ J^2_\omega = \mathbb{I},$$
$$\Delta_\omega^{-\frac{1}{2}} = J_\omega \Delta_\omega^{\frac{1}{2}} J_\omega,\ R^*_\omega = J_\omega \Delta_\omega^{-\frac{1}{2}}.$$
Moreover,
$$\begin{cases} \Delta^{it}_\omega \mathcal{N} \Delta^{-it}_\omega = \mathcal{N}, & \forall t \in \mathbb{R}, \\ J_\omega \mathcal{N} J_\omega = \mathcal{N}^\bullet. \end{cases}$$
[What about Takesaki theorem for KMS weights?]

[5] The tensor product of two algebraic states is defined as [[...]].

2.2 Relative modular theory

The description of the dependence of modular evolution on the change of algebraic state (or semi-finite weight) can be provided by Araki's ***relative modular operators*** $\Delta_{\omega',\omega}$ [1]. These operators can be defined by the unique polar decomposition $R_{\omega',\omega} = J_{\omega',\omega}\Delta^{1/2}_{\omega',\omega}$ of the closure of an operator $R_{\omega',\omega}$ defined by [30]

$$R_{\omega',\omega}A\Omega_\omega := A^*\Omega_{\omega'}, \tag{6}$$

where Ω_ω and $\Omega_{\omega'}$ are cyclic and separating vector representatives of the faithful normal states ω and ω' on the common Hilbert space $\mathcal{H}$. However, this holds only if the von Neumann algebra $\mathcal{N}$ is separable. The reformulation of definition (6) in more general terms is the following [2, 10, 33]. A subspace $\Pi \subset \mathcal{H}$ is called a ***cone*** iff $\forall\xi \in \Pi\ \forall\lambda \geq 0\ \lambda\xi \in \Pi$. A cone $\Pi \subset \mathcal{H}$ is called ***self-dual*** iff

$$\Pi = \{\eta \in \mathcal{H} \mid \langle\xi, \zeta\rangle \geq 0\ \forall\xi \in \Pi\}.$$

Every self-dual cone is closed. If $\mathcal{N}$ is a W^*-algebra, $\mathcal{H}$ is the Hilbert space, π is a non-degenerate faithful normal representation of $\mathcal{N}$ on $\mathcal{H}$, J is anti-unitary involution ($J^2 = \mathbb{I}$, $J^* = J$, $\langle J\xi, J\zeta\rangle = \langle\zeta, \xi\rangle\,\forall\zeta, \xi \in \mathcal{H}$) operator on $\mathcal{H}$, and Π is a closed self-dual cone in $\mathcal{H}$ (called ***natural positive cone***), then the quadruple $(\mathcal{H}, \pi, J, \Pi)$ is called ***a standard representation*** of $\mathcal{N}$ and the quadruple $(\mathcal{H}, \pi(\mathcal{N}), J, \Pi)$ is called ***a standard form*** of $\mathcal{N}$ iff

1. J generates an anti-linear $*$-isomorphism $j : \pi(\mathcal{N}) \ni a \mapsto j(a) := JaJ \in \pi(\mathcal{N})^\bullet$,
2. $\xi \in \Pi \Rightarrow J\xi = \xi$,
3. $(\xi \in \Pi,\ a \in \mathcal{N}) \Rightarrow \pi(a)J\pi(a)\xi \in \Pi$,
4. $a \in \mathfrak{Z}_\mathcal{N} \Rightarrow j(\pi(a)) = \pi(a)^*$.

If the elements of $\mathcal{N}$ are identified with the elements of $\pi(\mathcal{N})$ acting on $\mathcal{H}$, then $\mathcal{N}$ is called to be in standard form, or to ***act standartly*** on $\mathcal{H}$. If $\mathcal{N}$ acts standartly on $\mathcal{H}$, then the following properties hold:

1. for every positive normal functional there exists a unique corresponding vector representative in the natural positive cone:

$$\forall\phi \in \mathcal{N}^+_* \ \exists!\xi_\phi \in \Pi\ \forall a \in \mathcal{N}\ \ \phi(a) = \langle\xi_\phi, \pi(a)\xi_\phi\rangle_\mathcal{H}\,,$$

2. the map $\phi \mapsto \xi_\phi$ is norm continuous,
3. $\xi \in \Pi$ is cyclic for $\mathcal{N}$ iff $\xi \in \Pi$ is separating for $\mathcal{N}$ iff $\xi \in \Pi$ is cyclic for $\mathcal{N}^\bullet$ iff $\phi_\xi := \langle\xi, \cdot\xi\rangle$ is faithful for $\mathcal{N}$,
4. $\xi \in \Pi \Rightarrow (J\xi \in \Pi$ and $aj(a)\xi \in \Pi\ \forall a \in \mathcal{N})$,
5. $\phi \in \mathcal{N}^+_{*0} \Rightarrow \bigcup_{a\in\mathcal{N}}\{aj(a)\xi_\phi\}$ is dense in Π.

The nontrivial result of Haagerup [33] is that *every* W^*-algebra has a faithful representation π such that $\pi(\mathcal{N})$ is in standard form, and, moreover, this representation is unique up to unitary equivalence. If $\mathcal{N}$ is separable, then it admits a faithful normal algebraic state, which provides the faithful normal GNS representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$, together with the Tomita–Takesaki modular conjugation J_ω and a closed self-dual cone

$$\Pi_\omega := \overline{\bigcup_{A\in\pi_\omega(\mathcal{N})} \{AJ_\omega A\Omega_\omega\}}. \tag{7}$$

The cone Π_ω is convex, pointed ($\Pi_\omega \cap (-\Pi_\omega) = \{0\}$), it spans linearly $\mathcal{H}_\omega$, and it satisfies $\Delta_\omega^{it}\Pi_\omega = \Pi_\omega \forall t \in \mathbb{R}$.

For $\phi, \omega \in \mathcal{N}_*$, $a \in \mathcal{N}$, $\eta \in [\mathcal{N}\xi_\omega]^\perp$, where $[\mathcal{N}\xi_\omega]^\perp$ denotes the Hilbert space complement of the closure of the linear span of the action of (representation π of) the algebra $\mathcal{N}$ on the vector ξ, one defines the closeable anti-linear operator

$$R_{\phi,\omega}(a\xi_\omega + \eta) := \mathrm{supp}(\omega)a^*\xi_\phi,$$

acting on the dense domain $(\mathcal{N}\xi_\omega) \cup [\mathcal{N}\xi_\omega]^\perp \subset \mathcal{H}$. Its closure, denoted with the abuse of notation by the same letter, admits a unique polar decomposition

$$R_{\phi,\omega} = J\Delta_{\phi,\omega}^{1/2},$$

where J is the anti-unitary of the standard cone, while $\Delta_{\phi,\omega}$ is called a ***relative modular operator*** (if both Ω_ω and $\Omega_{\omega'}$ in (6) belong to natural positive cone, then $J_{\omega',\omega} = J$.). It satisfies

$$\Delta_{\phi,\omega} = R^*_{\phi,\omega}R_{\phi,\omega}, \quad \Delta_{\phi,\omega} = J\Delta^{-1}_{\omega,\phi}J, \quad \Delta_{\phi,\phi} = \Delta_\phi,$$
$$\Delta_{\lambda\phi,\mu\omega} = \tfrac{\lambda}{\mu}\Delta_{\phi,\omega}, \quad \forall\lambda,\mu \in \mathbb{R}.$$

The relative modular operators allow to define a continuous one-parameter family of unitary ***Connes' cocycle*** [9] of a pair (ω', ω) of faithful normal semi-finite weights on a von Neumann algebra $\mathcal{N}$

$$\mathbb{R} \ni t \mapsto \left(\frac{\mathrm{D}\omega'}{\mathrm{D}\omega}\right)_t := (\mathrm{D}\omega' : \mathrm{D}\omega)_t := \Delta^{it}_{\omega',\omega''}\Delta^{-it}_{\omega,\omega''} \in \mathcal{N},$$

where $\omega'' \in \mathcal{W}_0(\mathcal{N})$ is arbitrary. Connes cocycle always exists and it satisfies the following properties:

$$\begin{gathered}
\left(\tfrac{\mathrm{D}\omega_1}{\mathrm{D}\omega_2}\right)_0 = \mathbb{I}, \quad \left(\tfrac{\mathrm{D}\omega_1}{\mathrm{D}\omega_2}\right)_t \left(\tfrac{\mathrm{D}\omega_2}{\mathrm{D}\omega_3}\right)_t = \left(\tfrac{\mathrm{D}\omega_1}{\mathrm{D}\omega_3}\right)_t, \\
\left(\tfrac{\mathrm{D}\omega_1}{\mathrm{D}\omega_2}\right)^*_t = \left(\tfrac{\mathrm{D}\omega_2}{\mathrm{D}\omega_1}\right)_t, \quad \sigma_t^{\omega_1}(A) = \left(\tfrac{\mathrm{D}\omega_1}{\mathrm{D}\omega_2}\right)_t \sigma_t^{\omega_2}(A) \left(\tfrac{\mathrm{D}\omega_1}{\mathrm{D}\omega_2}\right)^*_t, \\
\left(\tfrac{\mathrm{D}\omega_1}{\mathrm{D}\omega_2}\right)_t (\sigma_t^{\omega_1}(A)) = (\sigma_t^{\omega_1}(A)) \left(\tfrac{\mathrm{D}\omega_1}{\mathrm{D}\omega_2}\right)_t, \\
\left(\tfrac{\mathrm{D}\omega_1}{\mathrm{D}\omega_2}\right)_{t_1+t_2} = \left(\tfrac{\mathrm{D}\omega_1}{\mathrm{D}\omega_2}\right)_{t_2} \sigma_{t_2}^{\omega_2}\left(\left(\tfrac{\mathrm{D}\omega_1}{\mathrm{D}\omega_2}\right)_{t_1}\right).
\end{gathered} \tag{8}$$

Moreover, for any unitary $U \in \mathcal{N}$ and any faithful normal semi-finite weights ω', ω on $\mathcal{N}$,

$$\omega'(\cdot) = \omega(U \cdot U^*) \iff \left(\frac{\mathrm{D}\omega'}{\mathrm{D}\omega}\right)_t = U^*\sigma_t^\omega(U).$$

For a detailed treatment of relative modular operators and Connes' cocycle see [3].

[Write about $\Delta(\phi/\varphi)$ as a predecessor of $\frac{\mathrm{d}\phi}{\mathrm{d}\varphi}$, and cite Sherstnev for ***lineal***.]

The ***Connes spatial derivative*** $\frac{\mathrm{d}\phi}{\mathrm{d}\varphi}$ is a generalisation of a relative modular operator [12]. If $\omega \in \mathcal{W}_0(\mathcal{N})$ and $\omega^\bullet \in \mathcal{W}_0(\mathcal{N}^\bullet)$, then $\frac{\mathrm{d}\omega}{\mathrm{d}\omega^\bullet}$ is uniquely defined as a largest positive self-adjoint linear operator T such that

$$\left\|T^{1/2}\xi\right\|^2 = \omega(R_{\omega^\bullet}(\xi)R_{\omega^\bullet}(\xi)^*),$$

for all

$$\xi \in \{\zeta \in \mathcal{H} \mid \exists c > 0\ \forall a \in \mathcal{N}^\bullet \ \ \|a\zeta\|^2 \leq c\omega^\bullet(a^*a), \}$$

and for a unique bounded linear operator

$$R_{\omega^\bullet}(\xi) : \mathcal{H}_{\omega^\bullet} \ni \pi_{\omega^\bullet}(a)\Omega_{\omega^\bullet} \mapsto a\xi \in \mathcal{H},$$

where $(\mathcal{H}_{\omega^\bullet}, \pi_{\omega^\bullet}, \Omega_{\omega^\bullet})$ is a GNS representation of $\mathcal{N}^\bullet$ with respect to $\omega^\bullet$. The map $\xi \mapsto R_{\omega^\bullet}(\xi)$ is linear and $R_{\omega^\bullet}(\xi)R_{\omega^\bullet}(\xi)^* \in \mathcal{N}^{\bullet\bullet} = \mathcal{N}$. In particular, if $\Omega \in \mathcal{H}$ is cyclic and separating for $\mathcal{N}$, and if $\forall a \in \mathcal{N}\ \forall a^\bullet \in \mathcal{N}^\bullet\ \omega(a) = \langle \Omega, a\Omega\rangle$ and $\omega^\bullet(a^\bullet) = \langle\Omega, a^\bullet\Omega\rangle$, then $\frac{\mathrm{d}\omega}{\mathrm{d}\omega^\bullet} = \Delta_\omega$.

2.3 Classification of factors

A ***trace*** is defined as a weight ω on $\mathcal{A}$ such that $\omega(a) = \omega(uau^*)\ \forall a \in \mathcal{A}^+$ and for all unitary $u \in \mathcal{A}$ (this is equivalent to the condition $\omega(aa^*) = \omega(a^*a)\ \forall a \in \mathcal{A}$). If a von Neumann algebra $\mathcal{N}$ is isomorphic to an algebra of a complex finite-dimensional matrices, then the example of a trace is given by the ordinary trace $A \mapsto \mathrm{tr}(A)$ of the matrix. If $\mathcal{N} = \mathfrak{B}(\mathcal{H})$, $\mathcal{H}$ is separable and $P_\mathcal{M} \in \mathcal{P}(\mathcal{N})$ is a projection operator on a subspace $\mathcal{M} \subseteq \mathcal{H}$, then [every?] trace tr on $P_\mathcal{M}$ satisfies $\mathrm{tr}(P_\mathcal{M}) = \dim \mathcal{M}$.

In a consequence of the von Neumann double commutant theorem [73], the set $\mathcal{P}(\mathcal{N}) := \{P \in \mathcal{N} \mid P^2 = P^*\}$ of all projectors of a von Neumann algebra $\mathcal{N}$ generates $\mathcal{N}$, what is a remarkable property, because the general C^*-algebras may even have no non-zero projectors. This result has led von Neumann and Murray to classification of von Neumann algebras $\mathcal{N}$ based on the analysis of the properties of $\mathcal{P}(\mathcal{N})$ [46, 47, 74, 48, 75]. In particular, the notion of trace on a von Neumann algebra allows to provide a classification of all von Neumann algebras which are factors, by evaluation it on a set $\mathcal{P}(\mathcal{N}) \subseteq \mathcal{N}$ of all projections generating $\mathcal{N}$. A factor von Neumann algebra $\mathcal{N}$ is called to be of one of the following types iff there exists the dimension function $\mathfrak{d}$ on $\mathcal{N}$ which is a trace taking the corresponding values on the elements of $\mathcal{P}(\mathcal{N})$:

- ***type I$_n$*** iff $\bigcup_{P\in\mathcal{P}(\mathcal{N})} \mathfrak{d}(P) = \{0, 1, 2, \ldots, n\}$,
- ***type I$_\infty$*** iff $\bigcup_{P\in\mathcal{P}(\mathcal{N})} \mathfrak{d}(P) = \{0, 1, 2, \ldots, \infty\}$,
- ***type II$_1$*** iff $\bigcup_{P\in\mathcal{P}(\mathcal{N})} \mathfrak{d}(P) = [0, 1]$,
- ***type II$_\infty$*** iff $\bigcup_{P\in\mathcal{P}(\mathcal{N})} \mathfrak{d}(P) = [0, \infty]$,
- ***type III*** iff $\bigcup_{P\in\mathcal{P}(\mathcal{N})} \mathfrak{d}(P) = \{0, \infty\}$.

The classification of factor plays and important role, because:

1) every factor von Neumann algebra possesses a unique (up to a multiplicative constant) trace such that the trace of non-zero projection is non-zero,
2) every factor von Neumann algebra is of one of above types,
3) due to von Neumann's theorem [75], every von Neumann algebra is uniquely isomorphic to a direct integral of factor von Neumann algebras (the construction of a direct integral of von Neumann algebras, given in [75], is similar to the construction of the direct integral of Hilbert spaces).

For type I_n and type II_1 the trace $\mathfrak{d}$ is finite, normal, and faithful; for type I_∞ and type II_∞ it is semi-finite, normal and faithful, while for type III there exists no normal, faithful and (finite or semi-finite) trace. A von Neumann algebra is called: ***semi-finite*** iff it admits a normal faithful semi-finite trace; ***finite*** iff it admits normal faithful finite trace. Hence, a von Neumann algebra is semi-finite iff it contains no type III factor.

A factor von Neumann algebra is called: ***discrete*** iff it is of type I; ***continuous*** iff it is of type II or III; ***purely infinite*** iff it is of type III. A von Neumann algebra is called ***semi-finite*** iff it does not contain any type III factor.

Every type I_n factor is isomorphic to the algebra of complex $n \times n$ matrices, while every type I_∞ factor is isomorphic to the algebra $\mathfrak{B}(\mathcal{H})$ on a separable Hilbert space. Every type II_∞ factor is a tensor product of some type I_∞ factor and some type II_1 factor. A factor von Neumann algebra is called ***hyperfinite*** iff it contains a sequence of finite-dimensional von Neumann algebras $\{\mathcal{N}_i\}$ such that $\{\bigcup_i \mathcal{N}_i\}$ is dense in $\mathcal{N}$ in weak-$*$ topology. There exists the unique hyperfinite factor von Neumann algebras of type I_n (for each n separately), I_∞, II_1, II_∞, and $\mathrm{III}_{\lambda\in]0,1]}$ (to be defined below). A unique hyperfinite type II_∞ factor is equal to a tensor product of the unique hyperfinite type I_∞ and type II_1 factors. Every type III factor is equal to a crossed product $\mathcal{N} \rtimes \mathbb{R}$ of a type II_∞ factor with $\mathbb{R}$ (in particular, the unique hyperfinite type III_1 factor is a crossed product of a unique hyperfinite type II_∞ factor with $\mathbb{R}$).

The GNS representation π_ω is irreducible iff ω is pure. If $\pi_\omega(\mathcal{A})$ is of type II or type III factor, then ω cannot be pure and there exists no unitarily isomorphic representation $\pi_{\omega_1}(\mathcal{A})$ such that a cyclic vector $\Omega_{\omega_1} \in \mathcal{H}_{\omega_1}$ would be unitarily transformed to some $\psi \in \mathcal{H}_\omega$ and Ω_{ω_1} would be a GNS representative of a pure algebraic state ω_1.

Recall that the space of ***trace class operators*** over a Hilbert space $\mathcal{H}$, denoted $\mathfrak{B}_1(\mathcal{H})$, is defined as the completion of the set

$$\{A \in \mathfrak{B}(\mathcal{H}) \mid \dim \operatorname{ran}(A) \leq 0\}$$

in the norm $\|A\|_1 := \operatorname{tr}(|\sqrt{A^* A}|)$. If $\mathcal{N}$ is a von Neumann algebra acting on $\mathcal{H}$ then

$$\begin{array}{c} \omega \in \mathcal{S}(\mathcal{N}) \cap \mathcal{N}_* \iff \exists! \rho \in \mathfrak{B}_1(\mathcal{H}) \text{ such that:} \\ \rho \geq 0,\ \operatorname{tr}\rho = 1,\ \omega(\cdot) = \operatorname{tr}(\rho\, \cdot). \end{array} \tag{9}$$

An example of algebraic state on a von Neumann algebra that is not normal is the one defined on the algebra $\mathfrak{B}(L_2(\mathbb{R}, \mathrm{d}x))$ by the [Dirac 'eigenstate']. If $\dim \mathcal{H} = \infty$, then there exist pure algebraic states ϕ on $\mathfrak{B}(\mathcal{H})$ such that there does not exist any $\xi_\phi \in \mathcal{H}$ such that $\phi(A) = \langle \xi_\phi, A\xi_\phi \rangle$ $\forall A \in \mathfrak{B}(\mathcal{H})$. If the GNS representation $\pi_\phi(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$ is type II or type III factor, and if $\omega \in \mathcal{S}(\mathcal{A})$ is a pure algebraic state, then there exists no $\xi_\omega \in \mathcal{H}_\phi$ such that $\omega(A) = \langle \xi_\omega, A\xi_\omega \rangle$ $\forall A \in \pi_\phi(A)$.

The modular automorphism plays a crucial role in the structure of type III factors, because the algebraic states $\omega(\cdot)$ related with these factors cannot be represented in the form of trace $\mathrm{tr}(\rho\ \cdot)$ with some density operator ρ **[but this seems to be in contradiction with (9)!]**. In this case, the equation $\mathrm{tr}(\rho xy) = \mathrm{tr}(\rho yx)$ is meaningless and $\omega(xy) = \omega(yx)$ does not hold. However, this equation can be replaced by more general form $\omega(xy) = \omega(y\sigma_i^\omega(x))$, where σ_i^ω is a modular automorphism σ^ω with $t = i$, provided $\sigma_{t=0}^\omega = \mathrm{id}$. In this sense, the modular automorphisms measure the amount of non-traciality of the algebraic states.

In order to understand more precisely the meaning of modular evolution, we have to consider its relationship with the classification of factors. Let us define first, for any C^*-algebra $\mathcal{A}$, the group $\mathrm{Aut}\mathcal{A}$ of all automorphisms of $\mathcal{A}$, and the group $\mathrm{Int}\mathcal{A}$ of all ***inner automorphisms*** of $\mathcal{A}$, which are defined as such $\alpha \in \mathrm{Aut}\mathcal{A}$ that there exists a unitary operator $U \in \mathcal{A}$ satisfying

$$\alpha(A) = UAU^* \ \forall A \in \mathcal{A}.$$

Although $U_\omega(t)AU_\omega^*(t) \in \mathcal{N}$ $\forall t \in \mathbb{R}$, the relationship $U_\omega(t) \in \mathcal{N}$ holds only if $\mathcal{N}$ is semi-finite. Hence, consideration of type III factors implies the presence of the non-inner modular automorphisms (that is, $\Delta_\omega^{it} \notin \mathcal{N}$). Connes [9] has shown that the modular automorphisms classify completely the $*$-automorphisms of factor von Neumann algebras, in the sense that the non-inner modular automorphisms are the *only* non-inner $*$-automorphisms of factor von Neumann algebras.

Consider the spectrum $\mathrm{sp}(\Delta_\omega)$ of the modular operator Δ_ω and the ***modular spectrum*** $\mathfrak{sp}(\mathcal{N})$ of a von Neumann algebra $\mathcal{N}$, defined by

$$\mathfrak{sp}(\mathcal{N}) := \bigcap_{\omega \in \mathcal{W}_0(\mathcal{N})} \mathrm{sp}(\Delta_\omega).$$

The spectrum $\mathrm{sp}(\Delta_\omega)$ measures the periodicity of the modular automorphism group $\sigma_t^\omega = \mathrm{ad}\Delta_\omega^{it}$. If $\mathrm{sp}(\Delta_\omega) = \{1\}$ then σ_t^ω is equal to identity for every $t \in \mathbb{R}$. The modular spectrum $\mathfrak{sp}(\mathcal{N})$ is an algebraic invariant characterising the von Neumann factor. If $\mathcal{N}$ is a factor of type I or II, then $\mathfrak{sp}(\mathcal{N}) = \{1\}$. If $\mathcal{N}$ is a type III factor, then it is called

- ***type III***$_0$ iff $\mathfrak{sp}(\mathcal{N}) = \{0, 1\}$,
- ***type III***$_\lambda$ iff $\mathfrak{sp}(\mathcal{N}) = \{0\} \cup \{\lambda^n \mid n \in \mathbb{Z}, \lambda \in]0,1[\}$,
- ***type III***$_1$ iff $\mathfrak{sp}(\mathcal{N}) = [0, \infty[$.

Moreover, if $\mathcal{N}$ possesses a cyclic and separating vector Ω, then the conditions

- $\mathfrak{sp}(\mathcal{N}) = \{1\}$,
- $\sigma_t^\omega \in \mathrm{Int}\,\mathcal{N} \quad \forall t \in \mathbb{R}$,
- $\mathcal{N}$ is semi-finite,

are equivalent. This implies that for non-type III factors the modular automorphism reduces to inner unitary automorphism, while for type III factors it *is not* inner, that is, unitary Δ_ω^{it}, induced by state ω, do not belong to $\mathcal{N}$. It is then a remarkable fact that for von Neumann factors the modular automorphisms are the only possible non-inner automorphisms. More precisely, the modular automorphisms define a canonical homomorphism

$$\sigma : \mathbb{R} \to \mathrm{Out}\mathcal{A} := \mathrm{Aut}\mathcal{A}/\mathrm{Int}\mathcal{A}.$$

3 Automorphisms and their representations

Within the frames of the Hilbert space based approach to mathematical foundations of quantum theory the temporal behaviour of quantum models is described primarily in terms of one-parameter group (or semi-group) of unitary linear operators in the Hilbert space.[6] Specification of this unitary group, and, more generally, specification of any group of unitary symmetries of a given quantum theoretic model, is provided usually by specification of an infinitesimal generator of this group. If a given one-parameter group of unitary linear operators is considered as providing the description of temporal evolution, then its self-adjoint generator is called a hamiltonian. It is specified either in algebraic way, as a part of algebra under consideration, and only later equipped with a specific quantitative representation (which is quite often built upon the properties of a given hamiltonian or all given algebra), or it is postulated directly in the quantitative form within the given fixed representation.

It is important to note that the temporal behaviour of quantum theoretic model that is given by the group of one-parameter unitary operators obtains a definite quantitative character only

1) by the use of spectral measure that bijectively corresponds to the generators of this group,

2) by passing to a spectral representation associated with this group.

But the choice of particular spectral representation depends on the additional choice of the density operator which (together with a spectral measure) defines the probability measure underlying this representation. This means that the quantitative description of temporal behaviour of quantum theoretic models depends crucially not only on the choice of hamiltonian, but also on the choice of density operator.

3.1 C^*-automorphisms and derivations

In principle, the main shift provided by an algebraic approach is the change in emphasis from the abstract Hilbert space $\mathcal{H}$ to the abstract C^*-algebra $\mathcal{A}$, which leads to consideration of $*$-automorphisms $\alpha \in \mathrm{Aut}\mathcal{A}$ and representations $G \ni g \mapsto \alpha_g \in \mathrm{Aut}\mathcal{A}$ of groups G in the role of main mathematical objects used for description of linear temporal evolution

[6] The arbitrary *decision* to consider the unitary Schrödinger type temporal evolution as having primary importance, and to consider the non-unitary von Neumann–Lüders type temporal evolution as having secondary importance is the reason why the latter and not the former is considered as 'problematic'.

and symmetries of the quantum models. The consideration of groups of $*$-automorphisms of a C^*-algebra instead of groups of unitary operators on the given Hilbert space is a non-trivial generalisation, because for the given $*$-automorphisms $\{\alpha_g\}_{g\in G}$ there might exist no unitary operators $U(g) \in \mathcal{A}$ such that $\alpha_g(A) = U(g)AU(g)^*\ \forall A \in \mathcal{A}$. Moreover, if such group exists, it might be non-unique.

The uniqueness might be guaranteed on the level of representation. If π_ω is the GNS representation of $\mathcal{A}$ in $\mathfrak{B}(\mathcal{H}_\omega)$ with a cyclic vector Ω_ω then there exists exactly one family $U(g)$ of unitary operators $U(g) : \mathcal{H}_\omega \to \mathcal{H}_\omega$ such that

$$\begin{cases} \pi_\omega(\alpha_g(a)) = U(g)\pi_\omega(a)U(g)^{-1}\ \forall a \in \mathcal{A} \\ U(g)\Omega_\omega = \Omega_\omega. \end{cases} \tag{10}$$

It is called a ***unitary representation*** of a group α_g.

Infinitesimal generators of groups of $*$-automorphisms are called *derivations*, and play the role of algebraic analogues of hamiltonians. A ***derivation*** of a C^*-algebra $\mathcal{A}$ is defined as a map $\eth : \mathrm{dom}(\eth) \to \mathcal{A}$, where $\mathrm{dom}(\eth) \subset \mathcal{A}$ is a *-subalgebra such that for every $\lambda_1, \lambda_2 \in \mathbb{C}$ and every $A, B \in \mathrm{dom}(\eth)$

1. $\eth(\lambda_1 A + \lambda_2 B) = \lambda_1\eth(A) + \lambda_2\eth(B)$,
2. $\eth(AB) = \eth(A)B + A\eth(B)$,
3. $\eth(A)^* = \eth(A^*)$.

This can be understood as a definition of an algebraic derivative (differential) of a $*$-automorphism of $\mathcal{A}$. In order to equip derivations with the definite quantitative meaning, one needs to relate them to the properties of operators in a concrete representation (namely, the properties of self-adjoint hamiltonian operators generating the corresponding unitary implementations of these $*$-automorphisms). However, this can be done uniquely only if the derivation is bounded and if the $*$-automorphism is norm continuous,[7] which puts too strong limitation on possible particular quantitative models that can be specified uniquely with the help of derivations [6, 54]. This leads to weakening of the continuity properties of $*$-automorphisms, but then the notion of derivation, as determined only by an algebra and $*$-automorphism, lacks enough structure in order to uniquely specify its

[7] For any C^*-algebra $\mathcal{A}$ one has $\mathrm{dom}(\eth) = \mathcal{A}$ iff $\eth$ is the generator of a norm continuous one-parameter group $\mathbb{R} \ni t \mapsto \alpha_t \in \mathrm{Aut}\mathcal{A}$ of $*$-automorphisms of $\mathcal{A}$. If $\mathrm{dom}(\eth) = \mathcal{A}$, then $\eth$ is bounded and for any representation π of $\mathcal{A}$ there exists $H = H^* \in \pi(\mathcal{A})^{\bullet\bullet}$ such that

$$\pi(\alpha_t(A)) = e^{itH}\pi(A)e^{-itH}\ \ \forall A \in \mathcal{A}\ \forall t \in \mathbb{R}.$$

In particular, if $\eth$ is a derivation on a W^*-algebra $\mathcal{N}$ and $\mathrm{dom}(\eth) = \mathcal{N}$, then $\eth$ is bounded and there exists $H = H^* \in \mathcal{N}$ such that

$$\eth(A) = i[H, A]\ \ \forall A \in \mathcal{N}.$$

For example, if $\mathbb{R} \ni t \mapsto U(t)$ is a strongly continuous one-parameter group of unitary operators such that $U(t)\mathcal{N}U(t)^* \subseteq \mathcal{N}\ \forall t \in \mathbb{R}$, then the family of maps

$$\mathcal{N} \ni A \mapsto \alpha_t(A) := U(t)AU(t)^*$$

is a weak-* continuous group of $*$-automorphisms of $\mathcal{N}$ whose generator $\eth$ has the form $\eth(A) = i[H, A]$, where $H = H^*$ is a self-adjoint generator of the group $U(t)$.

quantitative representation. In order to provide a refined quantitative characterisation of the properties of temporal behaviour of quantum theoretic model using the hamiltonian operator that corresponds to a given $*$-automorphism, one needs to impose several restrictions on the derivation and to determine several additional properties of representation using conditions imposed on the algebraic states (see [6] for details). As we will see, the introduction of an algebraic state as a structural element used to determine the concrete temporal behaviour of the given quantum theoretic model becomes crucial.

For example [6], a given derivation $\eth$ of a C^*-algebra (or, respectively, a factor W^*-algebra) $\mathcal{A}$ corresponds uniquely to a strongly continuous (or, respectively, σ-weakly continuous) one-parameter group of $*$-automorphisms of $\mathcal{A}$ iff it satisfies the following conditions:

1. $\eth$ is densely defined and closed in the respective topology,
2. $\mathrm{dom}(\eth)$ is a unital $*$-subalgebra of $\mathcal{A}$,
3. $\eth(\mathbb{I}) = 0$,
4. $(\mathbb{I} + \lambda\eth)\mathrm{dom}(\eth) = \mathcal{A} \;\; \forall\lambda \in \mathbb{R}\setminus\{0\}$,
5. $(\mathbb{I} + \lambda\eth)\mathrm{dom}(\eth)^+ \subset \mathrm{dom}(\eth)^+$.

This provides a unique link between the derivation and the $*$-automorphism, but leaves open the issue of unique specification of a corresponding quantitative hamiltonian. This can be done in may different inequivalent ways. Let us present two of them.

If $\eth$ is a derivation of a C^*-algebra $\mathcal{A} \subseteq \mathfrak{B}(\mathcal{H})$, $\mathrm{dom}(\eth) \subseteq \mathcal{A}$ is a $*$-subalgebra, $\Omega \in \mathcal{H}$ is a unit vector cyclic for $\mathcal{A}$, and $\omega(\cdot) := \langle\Omega, \cdot\, \Omega\rangle$ is an algebraic state, then the condition

$$\exists c \geq 0 \; \forall A \in \mathrm{dom}(\eth) \;\; |\omega(\eth(A))|^2 \leq c\sqrt{\omega(A^*A) + \omega(AA^*)} \tag{11}$$

implies that there exists a (not necessarily unique) symmetric operator H on $\mathcal{H}$ such that

$$\begin{cases} \eth(A)\xi = i[H, A]\xi \;\; \forall\xi \in \mathrm{dom}(H) \; \forall A \in \mathrm{dom}(\eth), \\ \mathrm{dom}(\eth)\mathrm{dom}(H) \subseteq \mathrm{dom}(H). \end{cases} \tag{12}$$

For another example, assume that $\eth$ is a derivation of C^*-algebra $\mathcal{A}$ that is a generator of a strongly continuous $\alpha_t \in \mathrm{Aut}\,\mathcal{A}$, and let $\omega \in \mathcal{S}(\mathcal{A})$ satisfy $\omega(\eth(0)) = 0 \; \forall A \in \mathrm{dom}(\eth)$. Then

$$\begin{cases} \pi_\omega(\eth(A))\xi = i[H, \pi_\omega(A)]\xi \;\; \forall\xi \in \pi_\omega(\mathrm{dom}(\eth))\Omega_\omega, \\ \pi_\omega(\mathrm{dom}(\eth))\Omega_\omega \subseteq \mathrm{dom}(H), \end{cases}$$

where $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is a GNS representation of $\mathcal{A}$ generated by ω and H is a self-adjoint generator of a one-parameter group of unitary transformations $U(t)$ that uniquely corresponds to α_t by

$$\begin{cases} \pi_\omega(\alpha_t(A)) = U(t)\pi_\omega(A)U(t)^*, \\ U(t)\Omega_\omega = \Omega_\omega, \end{cases} \tag{13}$$

which holds for all $A \in \mathcal{A}$ and all $t \in \mathbb{R}$.

However, if the algebraic states become included in the characterisation of the quantitative generator of unitary temporal behaviour, then one has to note that there exists another self-adjoint operator that generates the quantitative representation of a given $*$-automorphism. This operator is called a liouvillean, and it is uniquely determined by a given triple of C^*-algebra $\mathcal{A}$, its $*$-automorphism $\tau \in \mathrm{Aut}\mathcal{A}$, and an algebraic state $\omega \in \mathcal{S}(\mathcal{A})$ or a standard representation $(\mathcal{H}, \pi, J, \Pi)$.[8] As opposed to hamiltonian, it has remarkably general uniqueness properties, which do not require the passage through derivations and additional analytic conditions that constrain derivation and algebraic state. In consequence, the algebraic approach renders the notion of a hamiltonian less relevant then the notion of a liouvillean.

The first idea about relationships and differences between liouvilleans and hamiltonians in their role of generators implementing the $*$-automorphisms of algebras can be obtained by consideration of the following examples:

1. If $\mathcal{N}$ is a von Neumann algebra on $\mathcal{H}$ with a cyclic vector Ω and if a derivation $\eth$ of $\mathcal{N}$ satisfying (12) is implemented by a self-adjoint (but not necessarily bounded) operator H such that $\Omega \in \mathrm{dom}(H)$ and $H\Omega = 0$, then the condition
$$e^{itH}\mathcal{N}e^{-itH} = \mathcal{N} \;\; \forall t \in \mathbb{R}$$
is equivalent to:
$$e^{itH} \in \mathcal{N} \;\; \forall t \in \mathbb{R}$$
if $H \geq 0$, or to
$$\Delta_\Omega^{is} H \Delta_\Omega^{-is} = H \;\; \forall s \in \mathbb{R},$$
if Ω is separating for $\mathcal{N}$, $\mathrm{dom}(\eth) = \{A \in \mathcal{N} \mid i[H, A] \in \mathcal{N}\}$, $\mathrm{dom}(\eth)\Omega \subseteq \mathcal{H}$ is a core subspace for H, and Δ_Ω is a modular operator associated with $(\mathcal{N}, \Omega)$.

2. Let the one-parameter group of unitary operators $U(t) := e^{itH} \in \mathcal{N}$ with self-adjoint H be implementation of the corresponding $*$-automorphism α on a von Neumann algebra $\mathcal{N}$. If $\mathcal{N}$ is in standard representation $(\mathcal{H}, \pi, J, \Pi)$, then there exists also another unitary group $V(t)$ associated with $*$-automorphism α, uniquely determined by the condition $V(t)\Pi \subseteq \Pi$, and satisfying
$$V(t)AV(t)^* = \alpha_t(A) = U(t)AU(t)^*,$$
but with $V(t) \neq U(t)$. [If a von Neumann algebra $\mathcal{N}$ is semi-finite] then the self-adjoint generator K of $V(t)$ is related to the self-adjoint generator H of $U(t)$ by
$$K = H - JHJ.$$
The generator K is an example of a liouvillean operator, defined below.

[8] This operator is not called a 'hamiltonian', because in general it can have a spectrum that is not bounded from any side, while the notion of 'hamiltonian' is usually understood as referring to a self-adjoint operator with non-negative, or at least bounded from below, spectrum.

3.2 Liouvilleans

In this subsection we will recall some notions and properties from the theory of liouvilleans [54, 19]. A ***representation of a locally compact topological group*** G in the group $\mathrm{Aut}\mathcal{A}$ of all $*$-automorphisms of a C^*-algebra $\mathcal{A}$ is a map $\alpha : G \ni g \mapsto \alpha(g) =: \alpha_g \in \mathrm{Aut}\mathcal{A}$ such that

1. $\alpha(e) = \mathrm{id}_{\mathcal{A}}$,
2. $\alpha(g_1) \circ \alpha(g_2) = \alpha(g_1 \circ g_2)$.

A triple $(\mathcal{A}, G, \alpha)$ is called: a ***C^*-dynamical system*** iff $\mathcal{A}$ is a C^*-algebra and the map $G \ni g \mapsto \alpha_g(A) \in \mathrm{Aut}\mathcal{A}$ is a representation of G that is continuous in the norm topology of $\mathcal{A}$ for each $A \in \mathcal{A}$ (this is called ***strong continuity***[9]); a ***W^*-dynamical system*** iff $\mathcal{A}$ is a W^*-algebra and the map $g \mapsto \alpha_g(A)$ is a representation of G that is continuous in the weak-$*$ topology of $\mathcal{A}$ (this is called ***weak-$*$ continuity***). For example, for the Hilbert space $\mathcal{H}$ and a self-adjoint hamiltonian operator H on $\mathcal{H}$, a triple $(\mathcal{N}, \mathbb{R}, \alpha)$, with $\mathcal{N} := \mathfrak{B}(\mathcal{H})$ and $\alpha_t(A) := e^{itH} A e^{-itH}$ is a C^*-dynamical system and a W^*-dynamical system.

If $(\mathcal{N}, \mathbb{R}, \alpha)$ is a W^*-dynamical system with $\mathcal{N}$ in standard form $(\mathcal{H}, \pi(\mathcal{N}), J, \Pi)$, then:

1. $\exists!$ one-parameter group of unitaries $U(t)$, $t \in \mathbb{R}$, such that
$$\pi(\alpha_t(\mathcal{N})) = U(t)\pi(\mathcal{N})U(t)^* \ \ \forall t \in \mathbb{R},$$
2. $\exists!$ self-adjoint operator L on $\mathcal{H}$, called ***standard liouvillean***, such that:
   i) $U(t) = e^{itL}$,
   ii) $e^{itL}\Pi \subset \Pi$.

The uniqueness above holds up to unitary equivalence. The operator L satisfies also

iii) $[J, L] = 0$,

iv) $e^{itL}\mathcal{N}^{\bullet}e^{-itL} = \mathcal{N}^{\bullet} \ \ \forall t \in \mathbb{R}$,

Note that the definition of standard liouvillean L does not depend on the algebraic state ω. It depends only on the notions of a W^*-dynamical system and a standard representation of W^*-algebra. However, if $\mathcal{N}$ is a separable W^*-algebra and ω is faithful and normal, then the standard representation π is, up to unitary equivalence, the GNS representation π_ω.

If ω is a faithful state on a W^*-algebra $\mathcal{N}$, and $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is a GNS representation, then Ω_ω is cyclic and separating for $\pi_\omega(\mathcal{N})$. From the Tomita–Takesaki theorem it follows that the modular automorphism

$$\sigma^\omega_t(A) := \Delta^{it}_\omega A \Delta^{-it}_\omega, \ \ A \in \pi(\mathcal{N})^{\bullet\bullet},$$

[9] A strongly continuous group $\mathrm{ad}U(t)$ of $*$-automorphisms of von Neumann algebra $\mathcal{N}$ is also ***uniformly continuous*** (or ***norm continuous***), that is, $\lim_{t\to 0} \|U(t) - \mathbb{I}\| = 0$.

forms a W^*-dynamical system $(\pi(\mathcal{N})^{\bullet\bullet}, \mathbb{R}, \sigma^\omega)$, and the modular hamiltonian is equal to standard liouvillean and reads

$$K = L_\omega = -\log \Delta_\omega.$$

The same holds also for $(\pi_\omega(\mathcal{N}), \mathbb{R}, \sigma^\omega)$ if ω is faithful and normal. In consequence, this result can be applied to the faithful KMS algebraic states. Let $(\mathcal{A}, \mathbb{R}, \alpha)$ be a W^*-dynamical system with an associated normal algebraic state ω and a normal form $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega, L_\omega)$, let $\tilde{\omega}(\cdot) := \langle \Omega_\omega, \cdot\, \Omega_\omega \rangle$ be the normal extension of ω to the von Neumann algebra $\pi_\omega(\mathcal{A})^{\bullet\bullet}$, and let $\tilde{\alpha}_t(A) := e^{itL_\omega} A e^{-itL_\omega}$ denote the $*$-automorphism induced by α on $\pi_\omega(\mathcal{A})^{\bullet\bullet}$. If there exists a $*$-subalgebra $\mathcal{B} \subset \mathcal{A}$ such that

1. $\pi_\omega(\mathcal{B})$ is σ-weakly dense in $\pi_\omega(\mathcal{A})$,
2. for $\beta > 0$ and $\forall A, B \in \mathcal{B}$ there exist functions $\mathbb{C} \ni z \mapsto \omega(\alpha_z(A)B) \in \mathbb{C}$ and $\mathbb{C} \ni z \mapsto \omega(A\alpha_z(B)) \in \mathbb{C}$ that are analytic, respectively, for $\mathrm{im}(z) \in]-\beta, 0[$ and $\mathrm{im}(z) \in]0, \beta[$, bounded and continuous in the closure of these domains, and satisfying
$$\omega(\alpha_t(A)B) = \omega(B\alpha_{t+i\beta}(A)) \;\; \forall t \in \mathbb{R} \, \forall A, B \in \mathcal{B},$$

then

1. $\tilde{\omega}$ is faithful and KMS for $\tilde{\alpha}$ and β,
2. $(\pi(\mathcal{A})^{\bullet\bullet}, \mathbb{R}, \tilde{\alpha}, \tilde{\omega})$ is a W^*-dynamical system with $\pi(\mathcal{A})^{\bullet\bullet}$ in standard form,
3. the modular operator of $\tilde{\omega}$ is given by $\Delta_{\tilde{\omega}} = e^{-\beta L}$,

with a self-adjoint operator L. By the theorem of Takesaki, if L_ω is a standard liouvillean of a modular automorphism associated with ω, then ω is KMS with respect to α and β iff L is the standard liouvillean of α and $L = \beta L_\omega$.

3.3 Crossed products and dynamical systems

A ***covariant representation of a*** C^****-dynamical system*** $(\mathcal{A}, G, \alpha)$ is defined as a triple $(\mathcal{H}, \pi, U)$, where $\mathcal{H}$ is a Hilbert space, $\pi : \mathcal{A} \to \mathfrak{B}(\mathcal{H})$ is a non-degenerate representation of $\mathcal{A}$ on $\mathcal{H}$ and $U : G \ni g \mapsto U(g) \in \mathfrak{B}(\mathcal{H})$ is a unitary representation of G on $\mathcal{H}$ that is continuous in norm topology of $\mathfrak{B}(\mathcal{H})$ and satisfies

$$\pi(\alpha(g)(A)) = U(g)\pi(A)U(g)^* \;\; \forall A \in \mathcal{A} \, \forall g \in G.$$

A ***covariant representation of a*** W^****-dynamical system*** $(\mathcal{N}, G, \alpha)$ is defined in the same way as a covariant representation of a C^*-dynamical system, but with an additional requirement that the representation $\pi : \mathcal{N} \to \mathfrak{B}(\mathcal{H})$ has to be normal. The notions of $*$-dynamical system, covariant representation and crossed product are very closely related: every C^*-dynamical system defines a unique corresponding C^*-crossed product, every W^*-dynamical system defines a unique corresponding crossed product, and non-degenerate representations of each crossed product are in one-to-one correspondence with the covariant representations of the original $*$-dynamical system. In what follows, we will restrict the discussion of this relationship only to W^*-dynamical systems and the associated crossed products.

Let $(\mathcal{A}, G, \alpha)$ be a C^*-dynamical system, let $\mathrm{d}g$ be a Haar measure on G, and let $M(g)$ be the [modular function] on G. The linear space $C(\mathcal{A}, G)$ of continuous functions $f : G \to \mathcal{A}$ with compact support can be equipped with the structure:

$$\begin{aligned}(xy)(g) &:= \int_G \mathrm{d}h x(h)\alpha(h)(y(h^{-1}g)),\\ x^*(g) &:= M(g)^{-1}\alpha(g)(x(g^{-1}))^*,\\ \|x\|_1 &:= \int_G \mathrm{d}h \|x(h)\|.\end{aligned}$$

The completion of this space in the topology of the norm $\|\cdot\|_1$ is a Banach *-algebra, denoted by $L_1(\mathcal{A}, G)$. This space can be equipped also with an alternative norm

$$\|x\| := \sup \|\pi(x)\|, \tag{14}$$

where π varies over all representations of $L_1(\mathcal{A}, G)$ in a given Hilbert space $\mathcal{H}$. The ***C^*-crossed product*** of $\mathcal{A}$ and G with respect to α is defined as the completion of $L_1(\mathcal{A}, G)$ in the norm (14), and is denoted by $\mathcal{A} \rtimes_\alpha G$.

Let $(\mathcal{N}, G, \alpha)$ be a W^*-dynamical system, let $\mathrm{d}g$ be a Haar measure on a locally compact topological group G, and $\mathcal{H}$ be a Hilbert space. The linear space $C(\mathcal{H}, G)$ of continuous functions $f : G \to \mathcal{H}$ with compact support can be equipped with the inner product

$$\begin{aligned} C(\mathcal{H}, G) \times C(\mathcal{H}, G) \ni (\xi_1, \xi_2) \mapsto \\ \mapsto \langle \xi_1, \xi_2 \rangle := \textstyle\int_G \mathrm{d}g \, \langle \xi_1(g), \xi_2(g) \rangle \in \mathbb{C}. \end{aligned}$$

The completion of $C(\mathcal{H}, G)$ in the topology of the norm defined by this inner product is a Hilbert space $L_2(\mathcal{H}, G, dg)$. The equations

$$\begin{aligned}(\pi_0(x)\xi)(h) &:= \alpha_h^{-1}(x)\xi(h),\\ (u_0(g)\xi)(h) &:= \xi(g^{-1}h),\end{aligned}$$

define a normal faithful representation π_0 of $\mathcal{N}$ in $\mathfrak{B}(\mathcal{H})$ and a unitary representation u_0 of G in $\mathcal{H}$ that satisfies the covariance equation

$$u_0(g)\pi_0(x)u_0(g)^* = \pi_0(\alpha_g(x)),$$

and is continuous in the norm topology. The ***crossed product*** of $\mathcal{N}$ by G with respect to α is defined as a von Neumann algebra on $L_2(\mathcal{H}, G, \mathrm{d}g)$ generated by $\pi_0(\mathcal{N})$ and $u_0(G)$ and is denoted by $\mathcal{N} \rtimes_\alpha G$. An example of such crossed product was given in the previous subsection, with $G = \mathbb{R}$ and $\alpha = \sigma^\omega$ (the modular automorphism group).

The one-to-one correspondence between non-degenerate representations of crossed products and covariant representations of corresponding *-dynamical systems is given as follows. If (H, π, U) is a covariant representation of a C^*-dynamical system $(\mathcal{A}, G, \alpha)$, then the corresponding representation $\tilde{\pi}$ of a C^*-crossed product $\mathcal{A} \rtimes_\alpha G$ is given uniquely by

$$\tilde{\pi}(x) = \int_G \mathrm{d}g \pi(x(g))U(g), \quad x \in C(\mathcal{A}, G).$$

If $\mathcal{N} \rtimes_\alpha G$ is a crossed product, then the corresponding covariant representation of the original W^*-dynamical system $(\mathcal{N}, G, \alpha)$ is explicitly given in the definition of the crossed product $\mathcal{N} \rtimes_\alpha G$, and is provided by the non-degenerate representation $(\mathcal{H}, \pi_0, \lambda)$:

$$\tilde{x} = \int_G \mathrm{d}g \pi_0(x(g))\lambda(g), \quad x \in C(\mathcal{H}, G).$$

An important change related by this shift of perspective is consideration of the symmetries encoded by $*$-automorphisms of a C^*-algebra not in terms of the invariance properties of the algebra, but in terms of the algebraic invariance properties of the space of algebraic states on this algebra. Recall that if α is a $*$-automorphism of $\mathcal{A}$, then an element $\omega \in \mathcal{S}(\mathcal{A})$ is called ***invariant*** with respect to α iff $\omega \circ \alpha = \omega$. The space of all algebraic states invariant with respect to α is denoted by $\mathcal{S}^\alpha(\mathcal{A})$. If $\alpha : G \ni g \mapsto \alpha(g) \in \mathrm{Aut}\mathcal{A}$ is a fixed representation of a group G, then the space $\mathcal{S}^\alpha(\mathcal{A})$ is denoted also by $\mathcal{S}^G(\mathcal{A})$. If $\alpha_g \equiv \alpha(g)$ is a representation of a group G and if $G \ni g \mapsto \omega(\alpha_g(A))$ is continuous $\forall A \in \mathcal{A}$, then $\mathcal{S}^G(\mathcal{A}) \subset \mathcal{A}^{\mathbf{B}}$ is a non-empty, convex and compact in weak-$*$ topology. This holds, in particular, for every W^*- and C^*- dynamical system.

A ***quantum dynamical system*** is defined as a quadruple $(\mathcal{A}, G, \alpha, \omega)$, where $\mathcal{A}$ is a C^*-algebra, G is a locally compact topological group, $\alpha : G \ni g \mapsto \alpha_g \in \mathrm{Aut}\mathcal{A}$, and

$$\omega \in \mathcal{S}^\alpha_c(\mathcal{A}) := \{\omega \in \mathcal{S}^\alpha(\mathcal{A}) \mid G \ni g \mapsto \omega(A^*\alpha_g(A)) \text{ is continuous } \forall A \in \mathcal{A}\}.$$

The space $\mathcal{S}^\alpha_c(\mathcal{A})$ is a convex subspace of $\mathcal{S}^\alpha(\mathcal{A})$, closed in norm topology. Every C^*-dynamical system is a quantum dynamical system for any $\omega \in \mathcal{S}^\alpha(\mathcal{A})$, and every W^*-dynamical system is a quantum dynamical system for any $\omega \in \mathcal{S}^\alpha(\mathcal{A}) \cap \mathcal{A}_*$. If $(\mathcal{A}, G, \alpha, \omega)$ is a quantum dynamical system with $G = \mathbb{R}$ and $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is an associated GNS representation, then $\exists!$ self-adjoint operator L_ω on $\mathcal{H}_\omega$ such that

$$\begin{cases} \pi_\omega(\alpha_t(A)) = e^{itL_\omega}\pi_\omega(A)e^{-itL_\omega} \quad \forall t \in \mathbb{R}\ \forall A \in \mathcal{A}, \\ L_\omega \Omega_\omega = 0. \end{cases}$$

This representation of a $*$-automorphism extends to the group of $*$-automorphisms of $\pi_\omega(\mathcal{A})^{\bullet\bullet}$, implemented by the same unitary operators e^{itL_ω}. A ***normal form*** of a quantum dynamical system $(\mathcal{A}, \mathbb{R}, \alpha, \omega)$ is defined as a quadruple $(\mathcal{H}, \pi, \Omega, L)$ such that

1. $\omega(A) = \langle \Omega, A\Omega \rangle$,
2. $\overline{\pi(A)^{\bullet\bullet}} = \mathcal{H}$,
3. $\pi(\alpha_t(A))^{\bullet\bullet} = e^{itL}\pi(A)^{\bullet\bullet}e^{-itL}$,
4. $L\Omega = 0$.

The GNS representation equips every quantum dynamical system with a normal form that is unique up to unitary equivalence. The operator L_ω of this normal form is called ***normal liouvillean***.

The notions of standard and normal liouvillean associated with a given quantum dynamical system coincide if the algebraic state ω is invariant with respect to the $*$-automorphism α. In such case, for any linear self-adjoint operator L on GNS Hilbert space $\mathcal{H}_\omega$, the conditions $L\Omega_\omega = 0$ (L is a natural liouvillean) and $e^{itL}\Pi \subset \Pi \ \forall t \in \mathbb{R}$ (L is standard liouvillean) are equivalent. More generally, if $(\mathcal{A}, \mathbb{R}, \alpha, \omega)$ is a quantum dynamical system with a normal form $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega, L_\omega)$ and if $\tilde{\omega}(\cdot) := \langle \Omega_\omega, \cdot\, \Omega_\omega \rangle$ is faithful on $\pi_\omega(\mathcal{A})^{\bullet\bullet}$, then $\pi_\omega(\mathcal{A})^{\bullet\bullet}$ is in standard form and the associated normal liouvillean of α is also a standard liouvillean.

For example, if there is given a Hilbert space $\mathcal{H}$ and a self-adjoint hamiltonian operator H on $\mathcal{H}$, then a triple $(\mathcal{N}, \mathbb{R}, \alpha)$, with $\mathcal{N} := \mathfrak{B}(\mathcal{H})$ and $\alpha_t(A) := e^{itH} A e^{-itH}$ is a C^*-dynamical system and W^*-dynamical system. Any given density operator ρ on $\mathcal{H}$ such that $e^{-itH}\rho e^{+itH} = \rho$ defines $\omega(\cdot) := \mathrm{tr}(\rho\cdot)$ which is a normal algebraic state invariant with respect to α. In consequence, the quadruple

$$(\mathcal{N}, G, \alpha, \omega) := (\mathfrak{B}(\mathcal{H}), \mathbb{R}, e^{itH} \cdot e^{-itH}, \mathrm{tr}(\rho\cdot)) \tag{15}$$

is a quantum dynamical system. If $\mathcal{H}$ is separable, then the associated standard liouvillean (equal to associated normal liouvillean) is given by

$$e^{itL}A = e^{itH} A e^{-itH|_{\overline{\mathrm{ran}(\rho)}}}, \quad A \in \mathfrak{B}(\mathcal{H}).$$

However, if H has purely continuous spectrum, then $\mathfrak{B}_1(\mathcal{H}) \cap \{H\}^\bullet = \emptyset$, hence there exists also no density operator satisfying $e^{-itH}\rho e^{+itH} = \rho$, and then (15) does not form a quantum dynamical system, so there exists no normal liouvillean associated to $(\mathcal{N}, G, \alpha, \omega)$, but the standard liouvillean still can be considered.

4 Non-commutative integration

The Radon–Bourbaki theory [55, 50, 5] of integration is equivalent under some weak conditions [4] to the Daniell–Stone abstract theory of integration on the commutative Stone vector lattices [16, 17, 18, 61, 62]. Moreover, while the theory of integration based on the notion of measure on space cannot be directly generalised to the case of non-commutative algebras, because it lacks the appropriate notion of space that could be subjected to evaluation based on measure. On the other hand, the direct non-commutative generalisation of the theory of the Daniell–Stone integrals on the commutative Stone vector lattices is possible, and it is provided by the theory of semi-finite normal weights on W^*-algebras.

The theory of integration on W^*-algebras has a long history, which can be divided roughly into three stages. Segal [57, 58, 59], Dixmier [20, 21], Dye [22], Ogasawara and Yoshinaga [51], Kunze [43], Stinespring [60], Nelson [49] and Yeadon [79] have developed a theory of integration on the semi-finite von Neumann algebras provided by faithful normal semi-finite traces. The extension of the non-commutative integration theory to the arbitrary von Neumann algebra $\mathcal{N}$ equipped with arbitrary normal faithful semi-finite weight ω became possible only after development of the Tomita–Takesaki modular theory [68, 69, 63] together with the Takesaki duality for crossed products of the von Neumann algebras by locally compact abelian groups [64]. First construction of the general theory of non-commutative integration for arbitrary von Neumann algebras was provided by Sherstnev and Trunov [?, ?, ?, ?, ?] (see also [?]). However, this theory allowed only the construction of non-commutative L_1 and L_2 spaces, with no clear extension to full theory of non-commutative L_p spaces. First construction of the general non-commutative

L_p spaces $L_p(\mathcal{N},\omega)$ was given by Haagerup [34] (see also [66]). In his approach, based on crossed products [64] and operator valued weights [35, 36], the non-commutative L_p spaces are represented as Banach spaces of closed densely defined operators, but acting on different Hilbert spaces for each p. This result was remarkable but not completely satisfactory. Later, Trunov [70, 71], Connes [12] and Hilsum [37], Araki and Masuda [3, 45], Kosaki [?] and Terp [67], Zolotaryov [80, ?], and Tikhonov [?] have developed several different constructions of non-commutative L_p spaces associated with $\mathcal{N}$ and ω.

In particular, Connes and Hilsum have defined $L_p(\mathcal{N},\omega^\bullet)$ spaces with respect to a faithful normal semi-finite weight $\omega^\bullet$ on $\mathcal{N}^\bullet$ as Banach spaces of closed densely defined operators acting on a single Hilbert space, Terp, Zolotaryov and Kosaki have defined $L_p(\mathcal{N},\omega)$ as functional analytic interpolation spaces between $\mathcal{N}_*$ and $\mathcal{N}$, Araki and Masuda have defined $L_p(\mathcal{N},\omega)$ using standard form of $\mathcal{N}$ on a single Hilbert space, while Yamagami and Izumi have considered the extension of the construction of $L_p(\mathcal{N},\omega)$ spaces for $p\in\mathbb{C}$. The equivalence of some of these approaches (reflected in the isometric isomorphisms between the non-commutative L_p spaces in particular formulations, see...), together with the discoveries of remarkable algebraic structure underlying the structure of non-commutative L_p spaces and its relationship with the crossed products by Woronowicz [77] and Yamagami [78], as well as the extension of the Kosaki–Terp approach to complex case by Izumi [38, 39, 40] have posed the problem of formulation of the canonical theory of non-commutative L_p spaces, which would unify all these approaches. This problem has been solved by Falcone and Takesaki [26], marking the third stage of development of the theory.

4.1 Measurable operators

For the purpose of the following discussion, let us recall some facts about polar decompositions of possibly unbounded operators. The closed densely defined linear operator $a:\mathrm{dom}(a)\to\mathcal{H}$ is called: ***self-adjoint*** iff $a=a^*$ and $\mathrm{dom}(a)=\mathrm{dom}(a)^*$; ***positive*** iff $\langle a\xi,\xi\rangle\geq 0\;\forall\xi\in\mathrm{dom}(a)$. If a is positive, then one denotes it by $a\geq 0$. If $a\geq 0$, then the operator

$$b:=\int_{\mathrm{sp}(a)}\mathrm{E}^a(\lambda)\sqrt{\lambda}$$

is a unique positive operator satisfying $b^2=a$, and is denoted by $b=:a^{1/2}$. For every closed densely defined linear operator $x:\mathrm{dom}(x)\to\mathcal{H}$, the operator x^*x is self-adjoint and positive, hence

$$|x|:=\sqrt{x^*x}=\int_{[0,\infty[}\mathrm{E}^{x^*x}(\lambda)\sqrt{\lambda}$$

is a unique positive operator satisfying $|x|^2=x$. Moreover, every closed densely defined operator x can be uniquely decomposed in a form $x=v|x|$, called a ***polar decomposition*** of x, where v is a partial isometry.

Let $\mathcal{N}$ be the semi-finite von Neumann algebra acting on $\mathcal{H}$ and let τ be a fixed faithful normal semi-finite trace on $\mathcal{N}$. A linear operator $x:\mathrm{dom}(x)\to\mathcal{H}$ is called ***affiliated*** to $\mathcal{N}$ iff $[x,u]=0$ for every unitary element $u\in\mathcal{N}^\bullet$. The space of all operators affiliated to $\mathcal{N}$ is denoted $\mathrm{aff}\mathcal{N}$. A closed densely defined linear operator $x:\mathrm{dom}(x)\to\mathcal{H}$ with polar decomposition $x = v|x|$ is affiliated to $\mathcal{N}$ iff any of the following equivalent conditions holds:

- $[u, x] = 0 \; \forall$ unitary $u \in \mathcal{N}^\bullet$,
- $[u, |x|] = 0$ and $[u, v] = 0 \; \forall$ unitary $u \in \mathcal{N}^\bullet$,
- $v \in \mathcal{N}$ and all spectral projections of $|x|$ belong to $\mathcal{N}$,

A closed densely defined linear operator $x : \mathrm{dom}(x) \to \mathcal{H}$ is called ***τ-measurable*** [58, 59] iff any of the following equivalent conditions holds:

- $\exists \lambda > 0 \;\; \tau(\mathrm{E}^{x^*x}([\lambda, +\infty[)) < \infty$,
- $\exists \; P = P^* = P^2 \in \mathcal{N}$ such that $\tau(\mathbb{I} - P) < \infty$, $P\mathcal{H} \subseteq \mathrm{dom}(x)$, and $\|xP\| < \infty$.

The space of all τ-measurable operators affiliated to $\mathcal{N}$ is denoted by $\mathfrak{N}(\tau)$. For $x, y \in \mathfrak{N}(\tau)$ the algebraic sum $x + y$ and algebraic product xy may not be closed, hence in general they do not belong to $\mathfrak{N}(\tau)$. However, their closures (denoted with the abuse of notation by the same symbol) do belong to $\mathfrak{N}(\tau)$. Moreover, $\mathbb{I} \in \mathfrak{N}(\tau)$. Hence, $\mathfrak{N}(\tau)$ is a complex unital $*$-algebra, with sum and multiplication defined by closures of algebraic sum and multiplication [49].

4.2 Haagerup's integration theory

Now we pass to Haagerup's theory. Let $\hat{\mathcal{N}} := \mathcal{N} \rtimes_{\sigma^\omega} \mathbb{R}$, where $\mathcal{N}$ is any von Neumann algebra, $\omega \in \mathcal{W}_0(\mathcal{N})$, while σ^ω is a modular automorphism group induced by ω on $\mathcal{N}$. The Takesaki duality for crossed products of von Neumann algebras [64] implies that $\hat{\mathcal{N}}$ is equipped with a one-parameter automorphism group $s \mapsto \hat{\sigma}^\omega_s$, $s \in \mathbb{R}$, which is dual to σ^ω on $\mathcal{N}$ in the sense that

$$(\mathcal{N} \rtimes_{\sigma^\omega} \mathbb{R}) \rtimes_{\hat{\sigma}^\omega} \mathbb{R} \cong \mathcal{N} \otimes \mathfrak{B}(L_2(\mathbb{R}, \mathrm{d}x)).$$

The von Neumann algebra $\mathcal{N}$ is a subalgebra of $\hat{\mathcal{N}}$, with an embedding $\mathcal{N} \to \hat{\mathcal{N}}$ characterised in terms of $\hat{\sigma}^\omega_s$ by

$$x \in \mathcal{N} \iff (\forall s \in \mathbb{R} \;\; \hat{\sigma}^\omega_s(x) = x) \;\; \forall x \in \hat{\mathcal{N}}.$$

The von Neumann algebra $\hat{\mathcal{N}}$ is always semi-finite, hence it can be equipped with a faithful normal semi-finite trace τ. A ***natural trace*** τ on $\hat{\mathcal{N}}$ is defined by

$$\tau \circ \hat{\sigma}^\omega_s = e^{-s} \tau. \tag{16}$$

The structure of $\hat{\mathcal{N}}$ is independent of the choice of ω *up to a $*$-isomorphism* preserving the group $\hat{\sigma}^\omega_s$ and the natural trace defined by (16) [72, 77]. These constructions allow to define the non-commutative L_p spaces $L_p(\mathcal{N}, \omega)$ associated with any given $\mathcal{N}$ and ω. Let τ denote the natural trace on $\hat{\mathcal{N}}$, associated with ω via $\hat{\sigma}^\omega_s$ and (16), and let $\mathfrak{M}(\hat{\mathcal{N}}, \tau)$ denote the completion of $\hat{\mathcal{N}}$ in τ-topology. This means, in particular, that if some Hilbert space $\mathcal{H}$ representing $\hat{\mathcal{N}}$ is chosen, $\mathfrak{M}(\hat{\mathcal{N}}, \tau)$ will be represented as a space of all closed densely defined τ-measurable operators affiliated with $\hat{\mathcal{N}}$. Let the extension of $\hat{\sigma}^\omega_s$, $s \in \mathbb{R}$, from $\hat{\mathcal{N}}$ to $\mathfrak{M}(\hat{\mathcal{N}}, \tau)$ be denoted, with the abuse of notation, by the same letter. Then the algebraic part of the structure of non-commutative $L_p(\mathcal{N}, \omega)$ space, where $p \in]0, \infty]$, is defined by

$$L_p(\mathcal{N}, \omega) := \{x \in \mathfrak{M}(\hat{\mathcal{N}}, \tau) \mid \hat{\sigma}^\omega_s(x) = e^{-\frac{s}{p}} x\}. \tag{17}$$

The space $L_\infty(\mathcal{N}, \omega)$ is defined as equal to $\mathcal{N}$. The space $\mathfrak{M}(\hat{\mathcal{N}}, \tau)$ is sometimes denoted by $L_0(\mathcal{N}, \omega)$, however we will not consider it as a non-commutative L_p space, but rather as a 'container' for all these spaces. Every space $L_p(\mathcal{N}, \omega)$ is a self-adjoint linear subspace of $\mathfrak{M}(\hat{\mathcal{N}}, \tau)$, closed under left and right multiplication by the elements of $\mathcal{N}$. If $x = v|x| \in \mathfrak{M}(\hat{\mathcal{N}}, \tau)$, then

$$x \in L_p(\mathcal{N}, \omega) \iff (v \in \mathcal{N}, \;\; |x| \in L_p(\mathcal{N}, \omega)).$$

All spaces $L_p(\mathcal{N}, \omega)$ inherit the τ-topology of $\mathfrak{M}(\hat{\mathcal{N}}, \tau)$, and all are sequential spaces with respect to it. Haagerup [32] has shown that there exists a linear bijection

$$\mathcal{N}_* \ni \phi \mapsto h_\phi \in L_1(\mathcal{N}, \omega) \tag{18}$$

preserving the positivity, conjugation, polar decomposition, and the action of $\mathcal{N}$. In particular, if $\phi = u|\phi|$ is a polar decomposition of $\phi \in \mathcal{N}_*$, considered as a linear form on $\mathcal{N}$, then $h_\phi = u h_{|\phi|}$ is a polar decomposition of h_ϕ considered as a closed densely defined operator. If a weight ϕ is normal semi-finite but unbounded, then $\phi \mapsto h_\phi$ defines a map to the space $\mathfrak{M}^1(\hat{\mathcal{N}}) \subset \mathfrak{M}(\hat{\mathcal{N}})$, where $\mathfrak{M}(\hat{\mathcal{N}})$ is a space of all closed densely defined operators affiliated with $\hat{\mathcal{N}}$ (which do not have to be τ-measurable), while $\mathfrak{M}^1(\hat{\mathcal{N}})$ is its subspace consisting of such $x \in \mathfrak{M}(\hat{\mathcal{N}})$ that satisfy

$$\hat{\sigma}^\omega_s(x) = e^{-s} x \;\; \forall s \in \mathbb{R}.$$

The space $L_1(\mathcal{N}, \omega)$ can be equipped with a bounded positive linear functional $\mathrm{Tr} : L_1(\mathcal{N}, \omega) \to \mathbb{C}$ such that

$$\forall \phi \in \mathcal{N}_* \;\; \mathrm{Tr}(h_\phi) = \phi(\mathbb{I}) = \|\phi\|_{\mathcal{N}_*}.$$

This allows to define the norm $\|\cdot\|_1$ on $L_1(\mathcal{N}, \omega)$ by the corresponding norm $\|\cdot\|_{\mathcal{N}_*}$ on $\mathcal{N}_*$:

$$\forall x \in L_1(\mathcal{N}, \omega) \;\; \|x\|_1 := \mathrm{Tr}(|x|) = \left\| h_\cdot^{-1}(|x|) \right\|_{\mathcal{N}_*}.$$

The Mazur map $\mathcal{N}^+ \ni x \mapsto x^p \in \mathcal{N}^+$, $p \in]0, \infty[$, extended by continuity to $\mathfrak{M}(\hat{\mathcal{N}}, \tau) \ni x \mapsto x^p \in \mathfrak{M}(\hat{\mathcal{N}}, \tau)$, enables one to define the corresponding norm (for $p \in [1, \infty[$) and p-norm (for $p \in]0, 1[$) by

$$\begin{aligned} &\forall x \in \mathfrak{M}(\hat{\mathcal{N}}, \tau)^+ \;\; x \in L_p(\mathcal{N}, \omega) \iff x^p \in L_1(\mathcal{N}, \omega), \\ &\forall x \in L_p(\mathcal{N}, \omega) \;\; \|x\|_p := \||x|^p\|_1^{1/p}, \end{aligned}$$

where $\mathfrak{M}(\hat{\mathcal{N}}, \tau)^+ := \{x^* x \mid x \in \mathfrak{M}(\hat{\mathcal{N}}, \tau)\}$ is a positive cone of $\mathfrak{M}(\hat{\mathcal{N}}, \tau)$. The ***Haagerup non-commutative*** L_p ***space*** is defined as the completion of $L_p(\mathcal{N}, \omega)$ in the topology generated by $\|\cdot\|_p$, and denoted with the abuse of notation, by the same symbol. For every $p \in [1, \infty[$, $L_p(\mathcal{N}, \omega) \subset \mathfrak{M}(\hat{\mathcal{N}}, \tau)$. For $\frac{1}{p} + \frac{1}{q} = 1$, $p \neq \infty$, the linear form

$$L_p(\mathcal{N}, \omega) \times L_q(\mathcal{N}, \omega) \ni (h, k) \mapsto \mathrm{Tr}(hk) = \mathrm{Tr}(kh) \in \mathbb{C} \tag{19}$$

defines the isometric duality pairing $[\![\cdot,\cdot]\!]$ between $L_p(\mathcal{N},\omega)$ and $L_q(\mathcal{N},\omega)$, given by

$$^{\mathbf{d}}: L_p(\mathcal{N},\omega) \ni x \mapsto x^{\mathbf{d}} := [\![x,\cdot]\!] := \mathrm{Tr}(x\,\cdot) \in L_q(\mathcal{N},\omega)^{\mathbf{d}}.$$

If $\mathcal{N}$ has no minimal projection and if $p < q$, then

$$\mathrm{Hom}(L_p(\mathcal{N},\omega), L_q(\mathcal{N},\omega)) = \{0\}.$$

The non-commutative Haagerup $L_p(\mathcal{N},\omega)$ space is isometrically isomorphic to a standard commutative Riesz space L_p only if $p = 2$ or if $\mathcal{N}$ is a commutative von Neumann algebra.

4.3 The Falcone–Takesaki core algebra

In what follows, we will often identify the W^*-algebra $\mathcal{N}$ with its standard representation von Neumann algebra $\pi(\mathcal{N})$ on the standard representation Hilbert space $\mathcal{H}(\mathcal{N})$ [33]. Every W^*-algebra $\mathcal{N}$ has a unique standard representation, up to a unitary isomorphism. This representation is faithful ($\ker(\pi) = \{0\}$) and it is unitarily isomorphic with the Gel'fand–Naĭmark–Segal representation [56] for a pair $(\mathcal{N},\omega)$, whenever $\omega \in \mathcal{N}^+_{*0}$ or $\omega \in \mathcal{W}_0(\mathcal{N})$. While the set $\mathcal{N}^+_{*0}$ is non-empty iff $\mathcal{N}$ is countably finite (i.e., it is isomorphic to a von Neumann algebra possessing a cyclic and separating vector), $\omega \in \mathcal{W}_0(\mathcal{N})$ exists for every W^*-algebra $\mathcal{N}$.

The crucial element of the Falcone–Takesaki theory [25, 24, 26], allowing further canonical construction of non-commutative $L_p(\mathcal{N})$ spaces, is the *core* von Neumann algebra $\widetilde{\mathcal{N}}$ associated functorially to any von Neumann algebra $\mathcal{N}$. The structure of $\widetilde{\mathcal{N}}$ is designed to include the elements of $\mathcal{N}$ and $\mathcal{W}_0(\mathcal{N})$ on equal footing[10], and it grew up as a refinement of the earlier results of van Daele [72], Woronowicz [77], Connes [11], and Yamagami [78].

For $x, x' \in \mathcal{N}$, $\omega, \omega' \in \mathcal{W}_0(\mathcal{N})$, one defines the equivalence relation $\sim_t$ on $\mathcal{N} \times \mathcal{W}_0(\mathcal{N})$ by

$$(x,\omega) \sim_t (x',\omega') \iff x' = x[\mathrm{D}\omega : \mathrm{D}\omega']_t. \tag{20}$$

The properties of Connes' cocycle (the chain rule) imply that (20) is an equivalence relation in $\mathcal{N} \times \mathcal{W}_0(\mathcal{N})$. The equivalence class $(\mathcal{N} \times \mathcal{W}_0(\mathcal{N}))/\sim_t$ is denoted by $\mathcal{N}(t)$, and its elements are denoted by $x\omega^{it}$. Definitions $(x\omega^{it} + y\omega^{it}) := (x+y)\omega^{it}$, $\lambda(x\omega^{it}) := (\lambda x)\omega^{it}$ for $\lambda \in \mathbb{C}$, and $\|x\omega^{it}\| := \|x\|$ equip $\mathcal{N}(t)$ with the structure of the Banach space, which is isometrically isomorphic to the Banach space structure of $\mathcal{N}$. One can translate between $\mathcal{N}(t)$ for different t's using the maps

$$\mathcal{N}(t) \times \mathcal{N}(t') \ni (x\omega^{it}, y\omega^{it'}) \mapsto x\sigma^\omega_t(y)\omega^{i(t+t')} \in \mathcal{N}(t+t'),$$
$$\mathcal{N}(t) \ni x\omega^{it} \mapsto \sigma^\omega_{-t}(x)^*\omega^{-it} \in \mathcal{N}(-t).$$

By definition, $\mathcal{N}(0)$ is isomorphic to $\mathcal{N}$ also as a von Neumann algebra. However, for $t \neq 0$ the spaces $\mathcal{N}(t)$ are not von Neumann algebras. The product topology on $\mathcal{N} \times \mathbb{R}$

[10] In order to obtain a rich theory of integration, one needs to guarantee the existence of inverse of the Radon–Nikodým quotient. The generalisation of the Radon–Nikodým quotient to the non-commutative setting is provided by the Connes cocycle, and its invertibility is guaranteed if the weight is faithful (from the perspective of the GNS representation, this condition implies that the Gel'fand ideal is empty) [9, 53].

(with $\mathcal{N}$ endowed with the weak-* topology or Arens–Mackey topology, but not with norm topology) allows to use the bijections $\mathcal{N}(t) \ni x\varphi^{it} \mapsto (x,t) \in \mathcal{N} \times \mathbb{R}$ to form Fell's Banach *-algebra bundle $F(\mathcal{N}) := \coprod_{t\in\mathbb{R}} \mathcal{N}(t)$ over $\mathcal{N} \times \mathbb{R}$ [27, 28].

This means that one can consider the Fell bundle $F(\mathcal{N})$ as a natural algebraic structure which enables to translate between elements of $\mathcal{N}(t)$ at different $t \in \mathbb{R}$. In order to recover an element of the Fell bundle at a given $t \in \mathbb{R}$, one has to select a section $\widetilde{x} : \mathbb{R} \to F(\mathcal{N})$ of $F(\mathcal{N})$:

$$t \mapsto x(t)\omega^{it} =: \widetilde{x}(t).$$

The set of sections of $F(\mathcal{N})$ can be equipped with a multiplication, involution, and norm,

$$\begin{aligned}(\widetilde{x}\widetilde{y})(t) &:= \textstyle\int_{\mathbb{R}} \mathrm{d}r x(r) y(t-r) = \left(\int_{\mathbb{R}} dr x(r) \sigma^{\omega}_r(y(t-r))\right)\omega^{it},\\ \widetilde{x}^*(t) &:= \widetilde{x}(-t)^* = \sigma^{\omega}_t(x(-t))^*\omega^{it},\\ \|\widetilde{x}\| &:= \textstyle\int_{\mathbb{R}} \mathrm{d}r \|x(r)\|,\end{aligned}$$

forming this way a Banach $*$-algebra, called the ***bundle algebra***, and denoted by $\mathcal{A}(\mathcal{N})$.

Our main interest now will be the action of this algebra on the suitably defined 'bundle of Hilbert spaces' over the line $\mathbb{R}$. The construction of this bundle resembles the construction of the bundle algebra, and is also based on a suitable equivalence relation. Consider a Hilbert space $\mathcal{H}$, line $\mathbb{R}$, and two von Neumann algebras $\mathcal{N}$ and $(\mathcal{N}^\bullet)^o$ related to $\mathcal{H}$. The algebra $(\mathcal{N}^\bullet)^o$ is an ***opposite algebra*** of $\mathcal{N}^\bullet$, defined as such von Neumann algebra, which has the same elements as $\mathcal{N}^\bullet$, but the opposite multiplication maps (that is, if $a, b, ab, a^* \in \mathcal{N}$, then $a^o, b^o, (ab)^o, (a^*)^o \in (\mathcal{N}^\bullet)^o$ and $(ab)^o = b^o a^o$, but $(a^*)^o = (a^o)^*$). The space $\mathcal{H}$ can be considered as a $\mathcal{N}$-$(\mathcal{N}^\bullet)^o$ bimodule, with the left action of $\mathcal{N}$ on $\mathcal{H}$ given by ordinary multiplication from left, and with the right action of $(\mathcal{N}^\bullet)^o$ on $\mathcal{H}$ defined by

$$\xi a^o := a\xi \quad \forall \xi \in \mathcal{H} \;\; \forall a^o \in (\mathcal{N}^\bullet)^o.$$

The equivalence relation

$$(r_1, \phi_1, u_1, \varphi_1) \quad \sim_t \quad (r_2, \phi_2, u_2, \varphi_2) \tag{21}$$

on the set $\mathbb{R} \times \mathcal{W}_0(\mathcal{N}) \times \mathcal{H} \times \mathcal{W}_0((\mathcal{N}^\bullet)^o)$ is defined by the condition

$$\left(\frac{\mathrm{d}\phi_1}{\mathrm{d}\varphi_1}\right)^{ir_1} u_1 = \left(\frac{\mathrm{d}\phi_2}{\mathrm{d}\varphi_2}\right)^{ir_2} u_2 \left(\frac{\mathrm{D}\varphi_2}{\mathrm{D}\varphi_1}\right)_t,$$

where $r_1, r_2 \in \mathbb{R}$, $u_1, u_2 \in \mathcal{H}$, $\phi_1, \phi_2 \in \mathcal{W}_0(\mathcal{N})$, and $\varphi_1, \varphi_2 : \mathcal{W}_0((\mathcal{N}^\bullet)^o)$.

The equivalence class of the relation (21) is denoted by $\mathcal{H}(t)$. Its elements have the form

$$\phi^{it}\xi = \left(\frac{\mathrm{d}\phi}{\mathrm{d}\varphi}\right)\xi\varphi^{it}.$$

Falcone and Takesaki have shown that $\mathcal{H}(t)$ is a Hilbert space independent of the choice of weights $\phi_1, \phi_2, \varphi_1, \varphi_2$ and of the choice of $r \in \mathbb{R}$. This enables to form the canonical Hilbert space bundle over $\mathbb{R}$,

$$\mathfrak{G} := \coprod_{t\in\mathbb{R}} \mathcal{H}(t),$$

and to form the Hilbert space of square-integrable cross-sections of $\mathfrak{G}$

$$\widetilde{\mathcal{H}} := \Gamma^2(\mathfrak{G}) = \Gamma^2(\coprod_{t\in\mathbb{R}} \mathcal{H}(t)).$$

The canonical Hilbert space bundle $\mathfrak{G}$ is homeomorphic to $\mathcal{H} \times \mathbb{R}$ for *any* choice of weight. The action of $\mathcal{A}(\mathcal{N})$ on $\widetilde{\mathcal{H}}$ generates the ***core von Neumann algebra*** $\widetilde{\mathcal{N}}$. For type III_1 factors $\mathcal{N}$ the core $\widetilde{\mathcal{N}}$ is a type II_∞ factor, but in general case $\widetilde{\mathcal{N}}$ is not a factor. Falcone and Takesaki have shown that the assignment $\mathcal{N} \mapsto \widetilde{\mathcal{N}}$ is functorial for any von Neumann algebra $\mathcal{N}$, and the structure of $\widetilde{\mathcal{N}}$ is independent of the choice of a weight on $\mathcal{N}$. However, for any choice of a weight ω on $\mathcal{N}$ there exists a unitary map

$$\mathfrak{u}_\omega : \widetilde{\mathcal{H}} = \Gamma^2(\coprod_{t\in\mathbb{R}} \mathcal{H}(t)) \to L_2(\mathcal{H}, \mathbb{R}, \mathrm{d}t) \cong \mathcal{H} \otimes L_2(\mathbb{R}, \mathrm{d}t),$$

such that

$$\mathfrak{u}_\omega(\xi)(t) = \phi^{-it}\xi(t) \in \mathcal{H} \quad \forall \xi \in \widetilde{\mathcal{H}}.$$

This map provides an isomorphism between the core von Neumann algebra $\widetilde{\mathcal{N}}$ on $\widetilde{\mathcal{H}}$ and the crossed product $\mathcal{N} \rtimes_{\sigma^\omega} \mathbb{R}$ on $\mathcal{H} \otimes L_2(\mathbb{R}, \mathrm{d}t)$, given by

...

The crossed product $\mathcal{N} \rtimes_{\sigma^\omega} \mathbb{R}$ acting on the Hilbert space $L_2(\mathcal{H}, \mathbb{R}, \mathrm{d}t) \cong \mathcal{H} \otimes L_2(\mathbb{R}, \mathrm{d}t)$ is defined as the von Neumann algebra generated by the operators $\pi_{\sigma^\omega}(x)$ and $u_\omega(t)$, which are defined by equations

$$(\pi_{\sigma^\omega}(x)\xi)(t) := \sigma^\omega_{-t}(x)\xi(t), \tag{22}$$

$$(u_\omega(t')\xi)(t) := \xi(t - t'), \tag{23}$$

where $x \in \mathcal{N}$, $t, t' \in \mathbb{R}$, $\xi \in L_2(\mathcal{H}, \mathbb{R}, \mathrm{d}t)$. These two operators satisfy the *covariance equation*

$$u_\omega(t)\pi_{\sigma^\omega}(x)u^*_\omega(t) = \pi_{\sigma^\omega}(\sigma^\omega_t(x)). \tag{24}$$

Rewriting the equation (23) as

$$(e^{-t'\frac{\mathrm{d}}{\mathrm{d}t}}\xi)(t) = \xi(t - t'),$$

we see that

$$u_\omega(t) = e^{it\widetilde{V}_\omega}, \tag{25}$$

where $\widetilde{V}_\omega := -i\frac{\mathrm{d}}{\mathrm{d}t}$. On the other hand, the equation (22) can be written as

$$(\pi_{\sigma^\omega}(x)\xi)(t) = \Delta^{it}_\omega x \Delta^{-it}_\omega \xi(t) = e^{-iK_\omega t} x e^{+iK_\omega t}\xi(t),$$

where K_ω is a modular hamiltonian of the modular operator Δ_ω. The covariance equation (24) uniquely translates the automorphism of the crossed product algebra $\mathcal{N} \rtimes_{\sigma^\omega} \mathbb{R}$ into the modular automorphism of the underlying von Neumann algebra $\mathcal{N}$:

$$e^{-it\widetilde{V}_\omega}\pi_{\sigma^\omega}(x)e^{it\widetilde{V}_\omega} = \pi_{\sigma^\omega}(e^{-itK_\omega} x e^{itK_\omega}). \tag{26}$$

Hence, there exists a bijective correspondence between the automorphism $\mathrm{ad}\, u_\omega(t)$ of the unitary representation $\mathcal{N} \rtimes_{\sigma^\omega} \mathbb{R}$ of the core $\widetilde{\mathcal{N}}$ and modular automorphism σ^ω_t of $\mathcal{N}$.

4.4 Non-commutative flow of weights

The one-parameter automorphism group of $F(\mathcal{N})$,

$$\tilde{\sigma}_s(x\phi^{it}) := e^{-its}x\phi^{it} \quad \forall x\phi^{it} \in \mathcal{N}(t), \tag{27}$$

extends uniquely to a group of automorphisms $\tilde{\sigma}_s : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}$. The automorphism $\tilde{\sigma}_t$ provides a weight-independent replacement for a dual automorphism $\hat{\sigma}^{\omega}_t$ used in Haagerup's theory. It allows to define a ***grade*** $\mathrm{grad}(R)$ of a closed densely defined operator R affiliated with $\widetilde{\mathcal{N}}$ as such $\gamma \in \mathbb{C}$ that

$$\tilde{\sigma}_s(R) = e^{-\gamma s}R \quad \forall s \in \mathbb{R}. \tag{28}$$

If $\mathrm{grad}(R) = 0$, then R is bounded, but if $\mathrm{re}\,\mathrm{grad}(R) \neq 0$, then R is unbounded. The action of $\tilde{\sigma}_s$ on $\widetilde{\mathcal{N}}$ is integrable over $s \in \mathbb{R}$, and

$$I_{\tilde{\sigma}}(x) := \int_{\mathbb{R}} \mathrm{d}s\tilde{\sigma}_s(x),$$

for $x \in \widetilde{\mathcal{N}}^+$, is an operator valued weight from $\widetilde{\mathcal{N}}$ to $\mathcal{N}$ (for details on operator valued weights, see [35, 36, 25]). This allows to equip $\widetilde{\mathcal{N}}$ with a faithful normal semi-finite trace $\tilde{\tau}$,

$$\tilde{\tau}_\varphi(x) := \lim_{\epsilon\to+0} \varphi \circ I_{\tilde{\sigma}}(\varphi^{-1/2}(1+\epsilon\varphi^{-1})^{-1/2}x\varphi^{-1/2}(1+\epsilon\varphi^{-1})^{-1/2}). \tag{29}$$

This definition is independent of the choice of weight ($\tilde{\tau}_\varphi = \tilde{\tau}_\psi \ \forall \varphi, \psi \in \mathcal{W}_0(\mathcal{N})$), which follows from the fact that

$$\left(\frac{\mathrm{D}\tilde{\tau}_\phi}{\mathrm{D}\tilde{\tau}_\varphi}\right)_t = \left(\frac{\mathrm{D}\tilde{\tau}_\phi}{\mathrm{D}\tilde{\varphi}}\right)_t \left(\frac{\mathrm{D}\tilde{\varphi}}{\mathrm{D}\tilde{\psi}}\right)_t \left(\frac{\mathrm{D}\tilde{\psi}}{\mathrm{D}\tilde{\tau}_\varphi}\right)_t = \varphi^{-it}\left(\frac{\mathrm{D}\varphi}{\mathrm{D}\psi}\right)_t \psi^{it} = 1 \ \ \forall \phi, \varphi \in \mathcal{W}_0(\mathcal{N}).$$

This allows to write $\tilde{\tau}$ instead of $\tilde{\tau}_\varphi$. Moreover, $\tilde{\tau}$ has the scaling property

$$\tilde{\tau} \circ \tilde{\sigma}_s = e^{-s}\tilde{\tau} \quad \forall s \in \mathbb{R}. \tag{30}$$

This allows to call $\tilde{\tau}$ a ***canonical trace*** of $\widetilde{\mathcal{N}}$.

Falcone and Takesaki call the system $(\widetilde{\mathcal{N}}, \mathbb{R}, \tilde{\sigma}, \tilde{\tau})$ the ***non-commutative flow of weights***. It defines a functor from the category **vNaIso** of von Neumann algebras ($\mathcal{N}$) with isomorphisms as morphisms to the category **FTflow** of semi-finite von Neumann algebras ($\widetilde{\mathcal{N}}$) equipped with one-parameter ($\mathbb{R}$) automorphism group ($\tilde{\sigma}$) scaling a faithful normal semi-finite trace ($\tilde{\tau}$) according to (30), with morphisms given by such isomorphisms which preserve the scaling property (30). The restriction of $\tilde{\sigma}$ to the center $\mathfrak{Z}_{\widetilde{\mathcal{N}}}$ of $\widetilde{\mathcal{N}}$ provides precisely the Connes–Takesaki flow of weights $(\mathfrak{Z}_{\widetilde{\mathcal{N}}}, \mathbb{R}, \tilde{\sigma})$, which is a functor to the category **CTflow** of orbits of semi-finite normal weights under the inner automorphisms group $\mathrm{Int}(\mathcal{M})$ (for details, see [14, 15, 13, 65]).

4.5 Non-commutative $L_p(\mathcal{N})$ spaces

Let $\mathfrak{M}^p(\widetilde{\mathcal{N}},\tilde{\tau})$ denote the space of all $\tilde{\tau}$-measurable operators of grade $1/p$ affiliated to $\widetilde{\mathcal{N}}$ (for details on measurable operators, see [58, 49, 66]), and let the set $\mathfrak{m}^+_{I_{\tilde{\sigma}}} \subseteq \widetilde{\mathcal{N}}^+$ be defined as

$$\mathfrak{m}^+_{I_{\tilde{\sigma}}} := \{x^*y \in \widetilde{\mathcal{N}}^+ \mid \|I_{\tilde{\sigma}}(x^*x)\| < \infty,\ \|I_{\tilde{\sigma}}(y^*y)\| < \infty\}. \tag{31}$$

Given a weight $\varphi \in \mathcal{W}_0(\mathcal{N})$ and a canonical trace $\tilde{\tau}$,

$$h^{it}_{\varphi} := [\mathrm{D}(\varphi \circ I_{\tilde{\sigma}}) : \mathrm{D}\tilde{\tau}]_t \ \ \forall t \in \mathbb{R} \tag{32}$$

defines operator valued maps $\varphi \mapsto h_\varphi$ on $\mathcal{N}^+_*$, called the Haagerup correspondence [34, 78]. From the Pedersen–Takesaki non-commutative generalisation [53] of the Radon–Nikodým theorem it follows that h_φ is a unique positive operator affiliated with $\widetilde{\mathcal{N}}$ that satisfies

$$\varphi \circ I_{\tilde{\sigma}}(\cdot) = \tilde{\tau}(h_\varphi\, \cdot). \tag{33}$$

Hence, h_φ can be considered as (a reference-independent) 'operator density' of $\varphi \in \mathcal{N}^+_*$. Moreover, for any closed and densely defined positive operator R affiliated with $\widetilde{\mathcal{N}}$ with $p := \mathrm{re}\,\mathrm{grad}(R) > 0$ there exists a unique weight φ such that h_φ is of grade 1 and $h_\varphi = |R|^{1/p}$. Given a polar decomposition of $\varphi = u|\varphi|$ as a linear form, the weight φ is finite ($|\varphi|(\mathbb{I}) < \infty$) iff h_φ is $\tilde{\tau}$-measurable [49, 34]. In consequence, an assignment $R(\varphi) := uR(|\varphi|)$ defines a unique extension of the Haagerup correspondence map to a natural bijection (linear isomorphism)

$$\mathcal{N}_* \ni \omega \mapsto R(\omega) \in \mathfrak{M}^1(\widetilde{\mathcal{N}},\tilde{\tau}). \tag{34}$$

Falcone and Takesaki show that the integral of $R \in \mathfrak{M}^1(\widetilde{\mathcal{N}},\tilde{\tau})$, given by

$$\int R := \tilde{\tau}(a^{1/2}Ra^{1/2}), \tag{35}$$

for any $a \in \mathfrak{m}^+_{I_{\tilde{\sigma}}}$ such that $I_{\tilde{\sigma}}(a) = 1$ is well defined. While $\tilde{\tau}$ takes only the $+\infty$ value on non-zero elements of $\mathfrak{M}^1(\widetilde{\mathcal{N}},\tilde{\tau})$, the integral $\int$ is finite. This allows to extend (34) to an isometric isomorphism of Banach spaces, with the norm on $\mathfrak{M}^1(\widetilde{\mathcal{N}},\tilde{\tau})$ defined by $\|R\|_1 := \int |R|$, and with $\|R(\varphi)\|_1 = \varphi(\mathbb{I})$. The duality pairing between Banach spaces $\mathcal{N}$ and $\mathfrak{M}^1(\widetilde{\mathcal{N}},\tilde{\tau})$ that identifies $\mathcal{N}_*$ with $\mathfrak{M}^1(\widetilde{\mathcal{N}},\tilde{\tau})$ is given by the bilinear form

$$\mathcal{N} \times \mathfrak{M}^1(\widetilde{\mathcal{N}},\tilde{\tau}) \ni (x, R) \mapsto [\![x, R]\!]_{\widetilde{\mathcal{N}}} := \int xR \in \mathbb{C}. \tag{36}$$

The non-commutative $L_p(\mathcal{N})$ spaces for $p \in [1,\infty[$ are defined as the spaces $\mathfrak{M}^p(\widetilde{\mathcal{N}})$ equipped with, and Cauchy complete in, the norm

$$\|R\|_p := \left(\int |R|^p\right)^{1/p}, \ \ R \in \mathfrak{M}^p(\widetilde{\mathcal{N}}). \tag{37}$$

By (34) and (36), $L_1(\mathcal{N}) \cong \mathcal{N}_*$, and it is natural to define $L_\infty(\mathcal{N}) \cong \mathcal{N} \cong \mathcal{N}(0)$, using the definition (28) of grade with $\tilde{\sigma}_s(R) = R$ for $\mathrm{grad}(R) = 0$. The space $L_2(\mathcal{N})$ is isometrically isomorphic to the Hilbert space $\mathcal{H}$ of standard representation of $\mathcal{N}$ [33], and the inner product on $L_2(\mathcal{N})$,

$$L_2(\mathcal{N}) \times L_2(\mathcal{N}) \ni (x_1, x_2) \mapsto \langle x_1, x_2 \rangle_{L_2(\mathcal{N})} := \int x_2^* x_1 \in \mathbb{C}, \tag{38}$$

allows their identification. For any choice of $\omega \in \mathcal{W}_0(\mathcal{N})$ (or $\omega \in \mathcal{N}_{*0}^+$), $L_2(\mathcal{N})$ is unitarily isomorphic to the GNS Hilbert space $\mathcal{H}_\omega$.

By definition, all $L_p(\mathcal{N})$ for $p \in [1, \infty]$ are Banach spaces. These spaces are isometrically isomorphic with the Kosaki–Terp, Araki–Masuda and Haagerup spaces $L_p(\mathcal{N}, \psi)$ and with the Connes–Hilsum spaces $L_p(\mathcal{N}, \psi^\bullet)$. The duality (36) extends to non-commutative $L_p(\mathcal{N})$ space duality, given by the bilinear map

$$L_p(\mathcal{N}) \times L_q(\mathcal{N}) \ni (S, R) \mapsto [\![S, R]\!]_{\widetilde{\mathcal{N}}} := \int SR \in \mathbb{C}, \tag{39}$$

with $1/p + 1/q = 1$, where $p \in \{z \in \mathbb{C} \mid \mathrm{re}\,(z) > 0\}$. For $p \in \mathbb{C}$ such that $\mathrm{re}\,(p) < 0$, one has $L_p(\mathcal{N}) = \{0\}$. This way the Falcone–Takesaki theory incorporates also Izumi's [38, 39, 40] complex extension of the weight-dependent Kosaki–Terp theory. Moreover,

$$L_{1/it}(\mathcal{N}) = \mathcal{N}(t) \;\; \forall t \in \mathbb{R}.$$

The spaces $L_p(\mathcal{N})$ are uniformly convex and uniformly smooth for $p \in]1, \infty[$. This follows from the isometric isomorphism with the Araki–Masuda and Kosaki–Terp $L_p(\mathcal{N}, \psi)$ spaces, which were shown to be uniformly convex and uniformly smooth in [] and [], respectively.

The spaces $\mathfrak{M}^p(\widetilde{\mathcal{N}}, \tilde{\tau})$, $p \in \mathbb{C} \setminus \{0\}$, embed naturally into the $*$-algebra $\mathfrak{M}(\widetilde{\mathcal{N}}, \tilde{\tau})$ of all $\tilde{\tau}$-measurable operators affiliated to $\widetilde{\mathcal{N}}$. Due to the properties of grade function, the elements of $\mathfrak{M}(\widetilde{\mathcal{N}}, \tilde{\tau})$ possess remarkable algebraic properties. The grade function satisfies:

$$\begin{gathered} \mathrm{grad}(R^*) = (\mathrm{grad}(R))^*, \\ \mathrm{grad}(\overline{SR}) = \mathrm{grad}(S) + \mathrm{grad}(R), \\ \mathrm{grad}(|R|) = \mathrm{re}\,(\mathrm{grad}(R)) = \tfrac{1}{2}(\mathrm{grad}(R) + \mathrm{grad}(R)^*), \\ \mathrm{re}\,(p) < 0 \Rightarrow L_{1/p}(\mathcal{N}) = \{0\}, \\ \mathrm{re}\,(\mathrm{grad}(R)) \geq 0 \Rightarrow |R|^{1/\mathrm{re}\,(\mathrm{grad}(R))} \in \mathcal{N}_*^+, \end{gathered} \tag{40}$$

where $\overline{SR}$ is the closure of SR, and the last property is understood via the bijection between $\omega \in \mathcal{N}_*^+$ and the corresponding $h_\omega \in L_1(\mathcal{N})$. If $\phi, \psi \in \mathcal{N}_*^+$ and $x, y \in \mathcal{N}$, then the elements $x\phi^\gamma$ and $y\psi^\lambda$ can be added and multiplied freely inside $\mathfrak{M}(\widetilde{\mathcal{N}}, \tilde{\tau})$ for all $\gamma, \lambda \in \mathbb{C}$ such that $\mathrm{re}\,(\gamma) \geq 0$ and $\mathrm{re}\,(\lambda) \geq 0$. This allows to use freely the algebraic operations on real and complex powers of weights, using the following identities: if $\{R_i\}_{i=1}^n \in \mathfrak{M}(\widetilde{\mathcal{N}}, \tilde{\tau})$ and $\sum_i \mathrm{grad}(R_i) = 1$ then

$$\begin{gathered} \prod_i R_i \in L_1(\mathcal{N}), \\ \int R_1 \cdots R_n = \int R_n R_1 \cdots R_{n-1}, \\ \|R_1 \cdots R_n\| \leq \|R_1\| \cdots \|R_n\|. \end{gathered}$$

Hence, in particular, one can use the formal algebraic relations of $\mathcal{N}(t)$, and rewrite the Connes cocycle as

$$\Delta^{it}_{\omega,\phi}\Delta^{-it}_{\phi} = \left(\frac{\mathrm{D}\omega}{\mathrm{D}\phi}\right)_t = \omega^{it}\phi^{-it}, \tag{41}$$

as well as rewrite the modular automorphism as

$$\Delta^{-it}_{\phi} x \Delta^{it}_{\phi} = \sigma^{\phi}_{-t}(x) = \phi^{-it}x\phi^{it}. \tag{42}$$

This enables to write an inner product

$$\mathcal{N}(t)\times\mathcal{N}(t) \ni (x\phi^{it}, y\phi^{it}) \mapsto \langle x\phi^{it}, y\phi^{it}\rangle := (y\phi^{it})^*(x\phi^{it}) = \phi^{-it}y^*x\phi^{it} = \sigma^{\phi}_t(y^*x) \in \mathcal{N}.$$

In the Falcone–Takesaki theory the trace $\tilde{\tau}$ and the algebra $\widetilde{\mathcal{N}}$ are canonical and independent of the choice of the particular weight on $\mathcal{N}$, as opposed to the trace τ and the algebra $\hat{\mathcal{N}}$ appearing in the Haagerup's theory. This makes the equation (34) a canonical generalisation of the equation (18). We will call the equation (34) the ***Segal–Haagerup–Falcone–Takesaki isomophism***. Because construction of $L_p(\mathcal{N})$ spaces as $\mathfrak{M}^p(\widetilde{\mathcal{N}},\tilde{\tau})$ is completely determined by the non-commutative flow of weights, the assignment of $L_p(\mathcal{N})$ spaces to von Neumann algebras $\mathcal{N}$ determines a functor $\mathbf{vNaIso} \to \mathbf{ncL}_p\mathbf{Iso}$, where $\mathbf{ncL}_p\mathbf{Iso}$ is a family of categories (indexed by $p \in [1,\infty]$) consisting of $L_p(\mathcal{N})$ spaces with isometric isomorphisms as morphisms.

Acknowledgements

I am indebted to prof. S.L. Woronowicz for valuable discussions that have introduced me to the world of modular theory and core algebra.

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