The Volume Operator in Loop Quantum Gravity

Johannes Brunnemann

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Overview

0 Motivation

1 Construction / Regularization / Implementation on $\mathcal{H}_{\text{kin}}$

2 Evaluation of Matrix Elements (Rep’n. Theory / Combinatorics)

3 Spectral Properties.

4 To Do.
Gauge Invariant 4-Vertex

- Gauge Invariance

\[ J_1 + J_2 + J_3 + J_4 \overset{!}{=} 0 \]

- Matrix Element

\[ \langle j_{12} | \hat{q}_{123} | j_{12} - 1 \rangle = \]
\[ = \frac{1}{\sqrt{(2j_{12} - 1)(2j_{12} + 1)}} \left[ (j_1 + j_2 + j_{12} + 1)(-j_1 + j_2 + j_{12})(j_1 - j_2 + j_{12})(j_1 + j_2 - j_{12} + 1) \right]^{\frac{1}{2}} \]
\[ \times (j_3 + j_4 + j_{12} + 1)(-j_3 + j_4 + j_{12})(j_3 - j_4 + j_{12})(j_3 + j_4 - j_{12} + 1) \]
\[ = - \langle j_{12} - 1 | \hat{q}_{123} | j_{12} \rangle \]
\[
< \bar{a} | \hat{a}_{IJK} | \bar{a}' > = \\
= \frac{1}{4} (-1)^{jK+jI+aI-1+aK} (-1)^{a_I-a'_I} (-1)^{\sum_{n=I+1}^{J-1} jn} (-1)^{-\sum_{p=J+1}^{K-1} jp} \times \\
\times X(j_I, j_J) \frac{1}{2} X(j_J, j_K) \frac{1}{2} \sqrt{(2a_I + 1)(2a'_I + 1)} \sqrt{(2a_J + 1)(2a'_J + 1)} \times \\
\times \left\{ \begin{array}{ccc}
\frac{a_I-1}{1} & j_I & a_I \\
 a'_I & j_I & I
\end{array} \right\} \left[ \prod_{n=I+1}^{J-1} \sqrt{(2a'_n + 1)(2a_n + 1)} (-1)^{a'_n-1+a_n-1+1} \left\{ \begin{array}{ccc}
j_n & a'_n-1 & a_n \\
 a_n & a_n & a_n-1
\end{array} \right\} \right] \times \\
\times \left[ \prod_{n=J+1}^{K-1} \sqrt{(2a'_n + 1)(2a_n + 1)} (-1)^{a'_n-1+a_n-1+1} \left\{ \begin{array}{ccc}
j_n & a'_n-1 & a_n \\
 a_n & a_n & a_n-1
\end{array} \right\} \right] \left\{ \begin{array}{ccc}
a_K & j_K & a_{K-1} \\
 a_{K-1} & a_{K-1} & a_{K-1}
\end{array} \right\} \times \\
\times \left[ (-1)^{a'_J+a'_J-1} \left\{ \begin{array}{ccc}
a_J & j_J & a'_J-1 \\
 a_J & a_{J-1} & \frac{1}{j_J}
\end{array} \right\} \left\{ \begin{array}{ccc}
a'_J-1 & j_J & a'_J \\
 a_J & a_J & \frac{1}{j_J}
\end{array} \right\} \\
- (-1)^{a_J+a_{J-1}} \left\{ \begin{array}{ccc}
a'_J & j_J & a'_J-1 \\
 a_J & a_{J-1} & \frac{1}{j_J}
\end{array} \right\} \left\{ \begin{array}{ccc}
a_{J-1} & j_J & a'_J \\
 a_J & a_J & \frac{1}{j_J}
\end{array} \right\} \right] \times \\
\times \prod_{n=2}^{I-1} a_n \delta_{a_n a'_n} \prod_{n=K}^{N} a_n \delta_{a_n a'_n} 
\]
4-Vertex
Analytical Insights

- Special Form: only 1 antisymmetric, $D$-dim tridiagonal matrix, sign factor $\sigma(123)$ only gives overall scaling of the spectrum.

$$\hat{q}_{123} = \begin{pmatrix}
0 & -q_1 & 0 & \cdots & 0 & 0 & 0 \\
q_1 & 0 & -q_2 & \cdots & 0 & 0 & 0 \\
0 & q_2 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & -q_{D-2} & 0 \\
0 & 0 & 0 & \cdots & q_{D-2} & 0 & -q_{D-1} \\
0 & 0 & 0 & \cdots & 0 & q_{D-1} & 0 \\
\end{pmatrix}$$

where $q_k = q_k(j_1, j_2, j_3, j_4)$
Numerical Results.

Histograms for the generic (gauge invariant) 4-vertex

... up to $j_{\text{max}} \leq 126/2$. (By ‘generic’ we mean excluding co-planar edges.)
Oriented Matroids
Motivation from Vectors I

$\mathbb{R}^3$, $M$ vector config with sorted ground set $E = (e_1, \ldots, e_5)$. 
Oriented Matroids
Motivation from Vectors I

$\mathbb{R}^3$, $\mathcal{M}$ vector config with sorted ground set $E = (e_1, \ldots, e_5)$. Characterized by linear dependence modulo
Oriented Matroids
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\[ \mathbb{R}^3, M \text{ vector config with sorted ground set } E = (e_1, \ldots, e_5). \]
Characterized by linear dependence modulo

(i) reorientation \( e_k \rightarrow -e_k \)
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$\mathbb{R}^3$, $\mathcal{M}$ vector config with sorted ground set $E = (e_1, \ldots, e_5)$. Characterized by linear dependence modulo

(i) reorientation $e_k \rightarrow -e_k$

(ii) re-labelling
Oriented Matroids
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$\mathbb{R}^3$, $\mathcal{M}$ vector config with sorted ground set $E = (e_1, \ldots, e_5)$.

Oriented Bases $\mathcal{B}(\mathcal{M})$
Oriented Matroids
Motivation from Vectors I

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- Family $\mathcal{B}(\mathcal{M})$ of sorted bases
  - $\mathcal{B} = \{ B = (b_1, b_2, b_3) \subseteq E : B \text{ spans } \mathbb{R}^3 \}$
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- Basis orientation $\chi_{\mathcal{B}}$ (chirotope), $S \subseteq E$
  - $\chi_{\mathcal{B}}(S) = \begin{cases} 
    \pm 1 & S \in \mathcal{B} \\
    0 & S \notin \mathcal{B} 
  \end{cases}$

... in our example $\chi_{\mathcal{B}}(B) = \pm \text{sgn}(|\det B|)$ (if $\chi_{\mathcal{B}}$ chirotope, then also $-\chi_{\mathcal{B}}$, depending of our notion of 'positive' orientation)
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\[ \mathbb{R}^3, \mathcal{M} \text{ vector config with sorted ground set } E = (e_1, \ldots, e_5). \]

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**Oriented Bases \( B(M) \)**

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  \[ B = \{ B = (b_1, b_2, b_3) \subseteq E : B \text{ spans } \mathbb{R}^3 \} \]

- **Basis orientation** \( \chi_B \) (chirotope), \( S \subseteq E \)

\[ \chi_B(S) = \begin{cases} 
\pm 1 & S \in B \\
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\end{cases} \]

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\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
B & 123 & 124 & 125 & 134 & 135 & 145 & 234 & 235 & 245 & 345 \\
\hline
\chi_B(B) & + & & & & & & & & & \\
\end{array}
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Oriented Matroids
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J. Brunnemann (HH / PB)  
\( \hat{V} \) in LQG  
3rd QG, Mar 3, 2011 7 / 20
\( \mathbb{R}^3, \mathcal{M} \) vector config with sorted ground set \( E = (e_1, \ldots, e_5) \).

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\hline
\chi_B(B) & + & + & 0 & - & - & - & + & + & + & 0 \\
\end{array}
\]
Oriented Matroids
Motivation from Vectors I

Re-labelling and reorientation act non-trivially on $\chi_B(B)$. One finds in total 4 (1 uniform) equivalence classes of chirotopes for $D = 3, N = 5$: 
### Oriented Matroids

**Motivation from Vectors I**

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<td>$\chi_{B,4}(B)$</td>
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Our example

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<tr>
<td>$\chi_B(B)$</td>
<td>+</td>
<td>+</td>
<td>0</td>
<td>−</td>
<td>−</td>
<td>−</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>0</td>
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is contained in equiv. class 3 (set $e_K \rightarrow -e_K$ for $K = 1, 3, 4, 5$ and use properties of $\text{det}$)
\( \mathbb{R}^3 \), \( \mathcal{M} \) vector config with sorted ground set \( E = \{ e_1, \ldots, e_5 \} \).
Oriented Matroids
Motivation from Vectors II

\[ \mathbb{R}^3, \mathcal{M} \text{ vector config with sorted ground set } E = \{e_1, \ldots, e_5\}. \]

Signed Circuits \( \mathcal{C} \)
Oriented Matroids
Motivation from Vectors II

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Signed Circuits \( \mathcal{C} \)

- **Circuits**
  - \( \mathcal{C} = \{ C \subseteq E : C \text{ min. lin. dep.} \} \)
  - Min. lin. dep. \( 0 = \sum_{k=1}^{N(c)} \lambda_K e_K \)
  - \( (e_K \in C, \lambda_K \in \mathbb{R}) \)
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Signed Circuits $\mathcal{C}$

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- Signed Subsets, $S \subseteq E$
  - $C = \{C^+, C^-\}$ where $C^\pm = \{e_K : \lambda_K \geq 0\}$
  - $(C')^\pm = C^\mp$. Both, $C, -C$ contained in $\mathcal{C}$
$\mathbb{R}^3$, $\mathcal{M}$ vector config with sorted ground set $E = \{e_1, \ldots, e_5\}$.

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... in our example $\mathcal{C} = \{\pm C_1, \pm C_2, \pm C_3\}$ is given by
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<table>
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<th>( C^+ )</th>
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<td>( { e_1, e_2, e_3 } )</td>
<td>( { e_4 } )</td>
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\( J. \text{ Brunnemann (HH / PB)} \)
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\[ J. \text{ Brunnemann (HH / PB)} \]
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  \[ \text{Relative Sign} \]
  \[ sgn_C(e_K) = \begin{cases} 
  \pm 1 & e_K \in C^\pm \\
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  \end{cases} \]
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        0 & e_K \notin C
    \end{cases} \]
  - \( \text{supp } C := C^+ \cup C^- \)
Oriented Matroids
Motivation from Vectors III

Description of vector config $\mathcal{M}$ over ground set $E$ in terms of $\mathcal{B}(\mathcal{M})$ and $\mathcal{C}(\mathcal{M})$ equivalent.

- for every $B \in \mathcal{B}$ and for every $e \in E \setminus B$ there is a unique $\pm C \in \mathcal{C}$ such that
  \[ B \cup \{e\} \subseteq C. \]

- Given two bases $B_1, B_2 \in \mathcal{B}$, $B_1 = (e, b_2, b_3)$, $B_2 = (f, b_2, b_3)$ we have $B_1 \cup \{f\} = B_2 \cup \{e\} \subseteq C$ for one $\pm C \in \mathcal{C}$. It holds that
  \[ \text{sgn}_C(e) \cdot \text{sgn}_C(f) = \chi_B(B_1) \cdot \chi_B(B_2). \]
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  \[ \text{sgn}_C(e) \cdot \text{sgn}_C(f) = \chi_B(B_1) \cdot \chi_B(B_2) \]

Can convert between the two equivalent descriptions!
Oriented Matroids
For Di-Graphs

The same combinatorics contained in a directed graph:
Oriented Matroids
For Di-Graphs

The same combinatorics contained in a directed graph:
Oriented Matroids
For Di-Graphs

The same combinatorics contained in a directed graph:

\[ M_{\text{vector}} \]

\[ M_{\text{graph}} \]

Signed Circuits \( C \) \( \equiv \) Loops
Oriented Matroids
For Di-Graphs

The same combinatorics contained in a directed graph:

Signed Bases $\mathcal{C}$  $\equiv$  Spanning Trees
Oriented Matroids
For Di-Graphs

The same combinatorics contained in a directed graph:

\[ \mathcal{M}_{\text{vector}} \]

\[ \mathcal{M}_{\text{graph}} \]

Signed Bases \( \mathcal{C} \) \( \equiv \) Spanning Trees
Oriented Matroids
For Di-Graphs

The same combinatorics contained in a directed graph:

$\mathcal{M}_{\text{vector}}$ $\mathcal{M}_{\text{graph}}$

$e_1$ $v_1$ $e_4$ $v_4$
$e_2$ $v_2$ $e_5$
$e_3$ $v_3$

Only two realizations of the more general combinatorial concept of an oriented matroid $\mathcal{M}$ of rank 3 over the ground set $E$ in terms of its signed bases $\mathcal{M} = (E, B)$, respectively signed circuits $\mathcal{M} = (E, C)$.  

J. Brunnemann (HH / PB)
Oriented Matroids

Axiomatic Definition: Signed Circuits

A family \( C \) of signed subsets of a finite set \( E \) is called the set of signed circuits of an oriented matroid \( \mathcal{M} = (E, C) \) on \( E \) if

(C0) Non-emptiness: \( \emptyset \notin C \)

(C1) Symmetry: \( C = -C \), that is for every \( C \in C \) also its opposite \(-C \in C\).

(C2) Incomparability: if \( C_1 \subseteq C_2 \) then either \( C_1 = C_2 \) or \( C_1 = -C_2 \)
\[ \forall C_1, C_2 \in C. \]

(C3) Elimination: For all \( C_1, C_2 \in C \) with \( C_1 \neq -C_2 \), if \( e \in C_1^+ \cap C_2^- \), \( \exists C_3 \in C \) such that \( C_3^\pm \subseteq (C_1^\pm \cup C_2^\pm) \setminus \{e\} \).

Equivalent formulation also in terms of \( B(\mathcal{M}) \). Can be extended to infinite ground sets [Bruhn et al.].
### More Difficult: Higher Valence

Sign factor combinatorics for 4–7-valent non-coplanar vertices

<table>
<thead>
<tr>
<th>$N_v$</th>
<th># triples</th>
<th>$# \vec{\epsilon}(N_v)$ sprinkled</th>
<th>$# \vec{\epsilon}$ perm. equiv. classes</th>
<th>$# \vec{\sigma}$ configs</th>
<th>$#$ realizable reor. equiv. classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>16</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
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<td>10</td>
<td>384</td>
<td>4</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>20</td>
<td>23,808</td>
<td>41</td>
<td>39</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>35</td>
<td>3,486,720</td>
<td>706</td>
<td>673</td>
<td>11</td>
</tr>
<tr>
<td>8</td>
<td>56</td>
<td>≥ 747,735,880</td>
<td>28,287</td>
<td>135</td>
<td>135</td>
</tr>
<tr>
<td>9</td>
<td>84</td>
<td>≥</td>
<td>?</td>
<td>?</td>
<td>4,381</td>
</tr>
</tbody>
</table>
Numerical Results.

Histograms for each sigma configuration $\vec{\sigma}$ at the (gauge invariant) 5-vertex

... up to $j_{\text{max}} = 25/2$. The blue is for $\vec{\sigma} = (\sigma_{123}, \sigma_{124}, \sigma_{134}, \sigma_{234}) = (2, 0, 0, 0)$, the green for $\vec{\sigma} = (2, 2, 2, 0)$, and the purple for $\vec{\sigma} = (2, 2, 4, 0)$. Each histogram has 512 bins.
Numerical Results

Histograms for the overall generic (gauge invariant) 5-vertex ... up to $j_{\text{max}} \leq 25/2$. (By ‘generic’ we mean excluding co-planar edges.) Each histogram has 512 bins.

![Histograms for the overall generic (gauge invariant) 5-vertex](image-url)
Numerical Results

Smallest non-zero eigenvalues $\lambda_{\text{min}}$ at the (gauge invariant) 5-vertex

\[ \lambda_{\text{min}} \]

Diagram showing $\lambda_{\text{min}}$ as a function of $2j_{\text{max}}$ and $\vec{\sigma}$-index.
Numerical Results

Largest eigenvalues $\lambda_{\text{max}}$ of the (gauge invariant) 5-vertex
T. Thiemann.  
*Modern Canonical Quantum General Relativity.*  

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