(III) DeSitter spacetime

1) DeSitter Basics
2) Free field DeSitter
3) Cosmic no hair
4) Interactions
5) Parametric rep. of DeSitter correlators
6) IR- stability

1) DeSitter basics:

DeSitter is a spacetime that:
- seems to describe early epochs of Universe
- seems to be recent (‘accelerated expansion’)

DeSitter space $dS_0$ can be introduced as

$$dS_0 = \{ x \in R^{D+1} | -x_0^2 + x_i^2 + \ldots + x_0^2 = H^{-2} \}$$

metric is that induced from $R^{D+1}$

$$ds^2 = -dt^2 + H^{-2} \cosh^2 t \cdot \omega_{D-1}^2$$

$H$ - Hubble constant

- DeSitter is a solution to Einstein eq.'s w/ positive cosm. const $\Lambda \propto H^2$
- DeSitter is a space of const. curvature ($\propto H^2$)
- Isometry group $O(D,1)$

A conformal diagram of $dS$ can be obtained by writing metric as $ds^2 = \Omega^{-2} [-dt^2 + \omega_{D-1}^2]$
In static chart $\frac{\partial}{\partial \eta}$ is a KVF

$\frac{\partial}{\partial \eta}$ is not globally time-like $\Rightarrow$ no analog of global "time-translations",

$\Rightarrow$
- No conserved energy for systems like e.g. $(\Box - m^2)\phi = 0$
- Notion of particle is problematic

\[ ds^2 = -f \, d\eta^2 + f^{-1} \, dr^2 + r^2 \, d\Omega^2 \]

Identify

\[ f(r) = 1 - H^2 r^2 \]

$\Rightarrow r = \pm H^{-1}$ on $\mathcal{H}_+^-$

\[ \mathbb{R}^{d-1} = \{t = \text{const.} \} \]

horizon \[ ds^2 = -dt^2 + e^{2Ht} \, dx^2 \]

"cosmological chart"

$ds_0 = \text{flat FRWL-spacetime}$

\[ (x_1, x_2) \in ds_0 \]

Let \[ Z = H^2 \cdot x_1 \cdot x_2 \]

"point-pair invariant"

$Z$ illustrates causal relationships in $ds_0$:

\[ \sigma = \text{cosine of timelike distance} \]

\[ \sigma = \text{cosine of spacelike distance} \]
2) The free field on $dS$ 

\[(\Box - m^2) \phi(x) = 0 \quad (\text{KG})\]

Want: States for such a quantum field

1) \(\langle \phi(x_1) \phi(x_2) \rangle_{\psi} \) has to satisfy (KG) in both \(x_1, x_2\)

2) \(\langle [\phi(x_1), \phi(x_2)] \rangle_{\psi} = 2i \text{Im} \langle \phi(x_1) \phi(x_2) \rangle = i \Delta(x_1, x_2) = \Delta_A - \Delta_R\)

3) Hadamard (OPE should hold)

4) Positive: \(\int_{x_1, x_2} \langle \phi(x_1) \phi(x_2) \rangle_{\psi} f(x_1) \overline{f(x_2)} \geq 0\) for any \(f\).

It is also natural to consider invariant states: 
\(\langle \phi(x_1) \phi(x_2) \rangle_{\psi} = \langle \phi(gx_1) \phi(gx_2) \rangle_{\psi} \quad \forall g \in O(d,1)\)

It turns out that there is a unique invariant state for \(m^2 > 0\):

\[\langle \phi(x_1) \phi(x_2) \rangle = \text{const.} \quad _2F_1 \left(\frac{-c, \frac{D-1+\epsilon}{2}; \frac{D-1}{2}}{\frac{1+Z}{2}} \right)\]

Bunch-Davies \(Z = H^2 x_1 \cdot x_2\)

\(\_2F_1\) hypergeom. fct.

\[c = -\frac{D-1}{2} + \sqrt{\left(\frac{D-1}{2}\right)^2 - \frac{m^2}{H^2}}\]

BD-state exists only for \(m^2 > 0\)

\[e = -(D-1) + i\beta\]

Relation between values of \(e\) and representation theory of \(SO(d,1)\)
\[ e^{-\frac{(d-1)c}{2}} \]

\[ \Phi = N_0 \]

\[ \text{Relationship between values of a \& representation theory of } SO(d,1) \]

1) \( \checkmark \)
2) \( \checkmark \)
3) Hadamard condition:

\[ \langle \Phi(x_1) \Phi(x_2) \rangle_{\text{BD}} = \text{cst.} \int \frac{\Gamma(0-1+c+2) \Gamma(-c+2) \Gamma(-2)}{\sin \pi \zeta} \frac{x}{\Gamma(\frac{d-1}{2}+\zeta)} dx \]

\[ \text{IR- poles} \quad \text{UV- poles} \]

Using residue thm for \( k' = 0 \):

\[ \langle \Phi(x_1) \Phi(x_2) \rangle_{\text{BD}} \sim \frac{1}{(1-2)^{1-d+2}} + \ldots \]

\[ \sim \frac{1}{\sigma^{d-1}} + \ldots \sigma \text{- geodesic distance} e \]

Hadamard cond. 3) \( \checkmark \)

Alternatively, deforming \( k \rightarrow k'' \& \) applying residue theorem

\[ \langle \Phi(x_1) \Phi(x_2) \rangle \sim \frac{1}{\cdot} \]
\[ \langle \phi(x_1) \phi(x_2) \rangle_{B_0} \sim \frac{1}{(1-2)^{\frac{D-1}{2}} + \text{i} \rho} + \cdots \]

as \( Z \to \infty \Rightarrow \langle \phi(x_1) \phi(x_2) \rangle_{B_0} \sim e^{-\frac{\mathcal{H}}{2} \tau} \]

\[ Z = \cosh \mathcal{H} \tau \]

A consequence of this IR behavior is that expectation value in arbitrary states decay exponentially in time

\[ |\psi\rangle = A |B_0\rangle \quad A = \int \prod_{x_i} p(x_1, \ldots, x_n) \phi(x_1) \cdots \phi(x_n) \]

\[ \Rightarrow \langle \phi(X) \rangle_\psi \sim O\left( e^{-\frac{H}{2} \tau} \right) \] where

\[ \tau \] is the time coordinate of \( X \)

\[ \langle \phi(x) \rangle_\psi \] decays exponentially

\[ \text{fix } |\psi\rangle \text{ down here} \]

\[ \Rightarrow \text{IR stability of deSitter spacetime} \]

4) Interactions

Purposes: 1) Give concise explicit formulas for

\[ \langle \phi(x_1) \cdots \phi(x_E) \rangle_{B_0, \lambda} \] in a theory like

\[ L = \left( \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2 - \lambda \phi^4 \right) d\mu \]
\[ t = \text{const.} \]

\[ R^3, \quad d s^2 = -d t^2 + e^{2Ht} d x^2 \]

\[ \langle \phi(t, \vec{x}_1) \ldots \phi(t, \vec{x}_E) \rangle \quad \text{important observables} \]

\text{in cosmology:}

i) \( E = 2 \quad \text{power spectrum of CMB} \)

ii) \( E = 3, \ldots \quad \text{Non-Gaussianities in CMB} \)

2) \( \text{IR stability in presence of } \lambda \phi^4 \)?

\[ \phi(x) = \text{interacting field} = \text{perturbative formula in terms of retarded products} \]

\[ dS_0^D = \left\{ X \in \mathbb{C}^{D+1} \mid X \cdot X = H^{-2} \right\} \]

contains both \( dS_0 \) but also \( S^D = D\)-sphere

Shortcut: Do my calculations of \( S^D \to \text{analytic continuation to } dS \)

Formula for correlation fcts.:

\[ \langle \phi(X_1) \ldots \phi(X_E) \rangle_{B^0, \lambda} = \text{anal. cont.} \left\{ \sum_{V = 0}^{\infty} \frac{(-\lambda)^V}{V!} \right\} \]

\[ \times \langle \phi(X_1) \ldots \phi(X_E) \left( \int_{S_0^D} \phi^\prime(Y) \, d\mu(Y) \right)^V \rangle_{B^0, \lambda} \]

= \text{anal. cont.} \sum_L c_L \lambda^L \mathcal{I}_G (X_1, \ldots, X_E)

Aim: Calculate \( \mathcal{I}_G \).

\[ X_0 \quad X_1 \quad X_2 \quad X_3 \quad X_4 \quad X_5, X_6, \ldots \quad \text{integrated over } S^D \]
\[ V = 8 \quad E = 4 \]

Propagator = \( \text{cst.} \, F_2 (-e, b-1+c; \frac{d}{2}; \frac{1+Z_{78}}{2}) \)

\[ Z_{78} = H^2 \chi_1 \chi_2 \quad (m^2 > 0) \]

\[ I_G = \int_{\text{internal vertices}} \prod \text{(propagators)} \]

\[ I_G (x_1, \ldots, x_E) = \left( \prod_{eE} \int_k d_2 e \right) \prod \text{(Gamma fct's of } 2_e) \]

\[ \prod_{\nu = E+1}^{V+E} \mu (x_0) \int_{S^0} \left[ (x_{i0} - x_{\nu0})^2 \right] \]

\[ \text{Need to do integrations over } S^0 \]

\[ \sim \int_{k} d_2 x_5 \int_{k} d_2 x_5 \cdots \] \( \Gamma^{'s} \)

\[ \int_{S^0} \mu (x_5) \int_{S^0} \mu (x_6) \]

\[ \left[ (x_i - x_5)^2 \right] \left[ (x_5 - x_6)^2 \right] \cdots \]

Renormalization

This is long story but it turns out that renormalization can be done in a reasonably explicit form.

\[ G \rightarrow G^x \]
Within \( G^* \) you consider trees

\[ T \subset G^* \] is a subgraph without loops

\[ \Gamma_G(\bar{w}) = \text{ext, } \int_{\bar{w}} \Gamma_G(\bar{w}) \prod_{1 \leq r < s \leq E} \left( 1 - Z_{rs} \right) \prod_{T \subset G^*} \frac{\sum w_T}{\text{conn. T W. S}} \]

- There is one contour integration over variable \( w \in \mathbb{C} \) for each tree \( T \subset G^* \), \( \bar{w} = \{ w_T \}_{T \subset G^*} \)

- \( \Gamma_G(\bar{w}) \) is a meromorphic fct of \( \bar{w} \)

- \( Z_{rs} = H^2 \chi_r \cdot \chi_s \)

\[ \Gamma_G(\bar{w}) = \frac{\Gamma\left( \frac{d+1}{2} + \sum w_T \right) \Gamma\left( -w_T \right) \prod_{(ij) \in T} \prod_{T' \supset (ij)} H\left( -\sum w_E \right)}{\Gamma\left( \frac{d+1}{2} - \sum w_T \right) \prod_{T' \subset E} \prod_{(ij) \in \Phi} \Gamma\left( \frac{d+1}{2} + \sum w_T \right) \prod_{T \ni (ij)}} \]

\( \Phi \) particular tree

IR- stability of \( ds_D \) in case of an interacting
IR-stability of $d_S$ in case of an interacting field:

How does $\langle \phi(x_1) \phi(x_2) \rangle_{b\pi, \lambda}$ behave for large time-like separations $x_1$, $x_2$?

$\Rightarrow$ How does $I_G(x_1, x_2)$ behave ... ?

$\Rightarrow$ Analysis of poles of $I_G$ shows that contours of $\tilde{w}$ can be moved to the left to achieve

$$\left(1-\frac{D}{2}\right)^{-\frac{1}{2}} \quad Re(...)<-\frac{D}{2}$$

$\Rightarrow I_G(x_1, x_2) \sim \left(\frac{2}{z_1 \cdot z_2}\right)^{-\frac{D}{2}}$ for $z_1 \rightarrow 0, z_2 \rightarrow 0$

$\Rightarrow \langle \phi(x_1) \phi(x_2) \rangle_{b\pi, \lambda} \sim O\left(e^{-H(z_1-z_2)}\right)$ as $z \rightarrow 0$

$\Rightarrow \langle \phi(x) \rangle_{\psi, \lambda} \sim O\left(e^{-H(z)}\right)$

$\Rightarrow$ Exponential decay of 1-pt fct. in arbitrary states

$$IC_{lm}^2 \sim \int \langle \phi(t, \vec{p}) \phi(t, \vec{0}) \rangle_{b\pi, \lambda} K_{lm}(\vec{k}) d^3k$$

$\Rightarrow$ Observable needs

$$\langle T_{\mu\nu}(X) \rangle_{b\pi, \lambda} = 0$$

$$\langle T_{\mu\nu}(X) \rangle_{\psi, \lambda} = ?$$